Fundamentals of Fluid Dynamics: Waves in Fluids

Introductory Course on Multiphysics Modelling

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(after: D.J. ACHESON's "*Elementary Fluid Dynamics*")

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1 Introduction

1.1 The notion of wave

What is a wave?

A **wave** is the **transport of a disturbance** (or energy, or piece of information) in space not associated with motion of the medium occupying this space as a whole. (Except that electromagnetic waves require no medium !!!)

- The transport is at **finite speed**.

- The shape or form of the **disturbance** is **arbitrary**.
- The disturbance moves with respect to the medium.

Two general classes of wave motion are distinguished:

1. **longitudinal waves** – the disturbance moves parallel to the direction of propagation. *Examples*: sound waves, compressional elastic waves (P-waves in geophysics);
2. **transverse waves** – the disturbance moves perpendicular to the direction of propagation. *Examples*: waves on a string or membrane, shear waves (S-waves in geophysics), water waves, electromagnetic waves.

1.2 Basic wave phenomena

reflection – change of wave direction from hitting a reflective surface,

refraction – change of wave direction from entering a new medium,

diffraction – wave circular spreading from entering a small hole (of the wavelength-comparable size), or wave bending around small obstacles,

interference – superposition of two waves that come into contact with each other,

dispersion – wave splitting up by frequency,

rectilinear propagation – the movement of light wave in a straight line.

Standing wave

A **standing wave**, also known as a **stationary wave**, is a wave that remains in a constant position. This phenomenon can occur:

- when the medium is moving in the opposite direction to the wave,
- (in a stationary medium:) as a result of interference between two waves travelling in opposite directions.

1.3 Mathematical description of a traveling wave

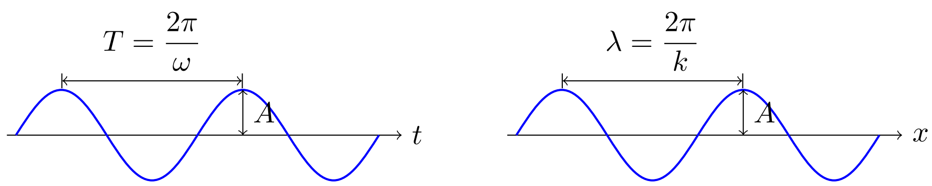


FIGURE 1: A simple traveling wave in time domain (*left*) and in space (*right*).

Traveling waves

Simple wave or **traveling wave**, sometimes also called *progressive wave*, is a disturbance that varies both with time t and distance x in the following way (see Figure 1):

$$\begin{aligned} u(x,t) &= A(x,t)\,\cos\left(k\,x - \omega\,t + \theta_0\right) \\ &= A(x,t)\,\sin\big(k\,x - \omega\,t + \underbrace{\theta_0 \pm \tfrac{\pi}{2}}_{\tilde{\theta}_0}\big) \end{aligned} \tag{1}$$

where A is the **amplitude**, ω and k denote the **angular frequency** and **wavenumber**, and θ_0 (or $\tilde{\theta}_0$) is the initial **phase**.

- **Amplitude** A $\left[\text{e.g. } \mathrm{m}, \mathrm{Pa}, \mathrm{V/m}\right]$ – a measure of the maximum disturbance in the medium during one wave cycle (the maximum distance from the highest point of the crest to the equilibrium).
- **Phase** $\theta = k\,x - \omega\,t + \theta_0$ $[\mathrm{rad}]$, where θ_0 is the *initial* phase (shift), often ambiguously, called the phase.
- **Period** T $[\mathrm{s}]$ – the time for one complete cycle for an oscillation of a wave.
- **Frequency** f $[\mathrm{Hz}]$ – the number of periods per unit time.

Frequency and angular frequency

The **frequency** f $[\mathrm{Hz}]$ represents the number of periods per unit time

$$f = \frac{1}{T}\,. \tag{2}$$

The **angular frequency** ω $[\mathrm{Hz}]$ represents the frequency in terms of radians per second. It is related to the frequency by

$$\omega = \frac{2\pi}{T} = 2\pi\,f\,. \tag{3}$$

- **Wavelength** λ $[\mathrm{m}]$ – the distance between two sequential crests (or troughs).

Wavenumber and angular wavenumber

The **wavenumber** is the spatial analogue of frequency, that is, it is the measurement of the number of repeating units of a propagating wave (the number of times a wave has the same phase) per unit of space.

> Application of a Fourier transformation on data as a function of time yields a **frequency spectrum**; application on data as a function of position yields a **wavenumber spectrum**.

The **angular wavenumber** k $\left[\frac{1}{\mathrm{m}}\right]$, often misleadingly abbreviated as "wave-number", is defined as

$$k = \frac{2\pi}{\lambda}\,. \tag{4}$$

There are two velocities that are associated with waves:

1. **Phase velocity** – the rate at which the wave propagates:

$$c = \frac{\omega}{k} = \lambda\, f\,. \tag{5}$$

2. **Group velocity** – the velocity at which variations in the shape of the wave's amplitude (known as the *modulation* or *envelope* of the wave) propagate through space:

$$c_{\mathsf{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}\,. \tag{6}$$

   This is (in most cases) the signal velocity of the waveform, that is, the **rate at which information or energy is transmitted** by the wave. However, if the wave is travelling through an absorptive medium, this does not always hold.

2 Water waves

2.1 Surface waves on deep water

- Consider **two-dimensional** water waves: $\boldsymbol{u} = \big[u(x, y, t),\ v(x, y, t),\ 0\big]\,.$
- Suppose that the flow is **irrotational**: $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0\,.$
- Therefore, there exists a **velocity potential** $\phi(x, y, t)$ so that

$$u = \frac{\partial \phi}{\partial x}\,, \qquad v = \frac{\partial \phi}{\partial y}\,. \tag{7}$$

- The fluid is **incompressible**, so by the virtue of the incompressibility condition, $\nabla \cdot \boldsymbol{u} = 0$, the velocity potential ϕ will satisfy **Laplace's equation**

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0\,. \tag{8}$$

Free surface

The fluid motion arises from a deformation of the water surface – which is of major interest (see Figure 2). The equation of this free surface is denoted by $\boxed{y = \eta(x, t)}$.

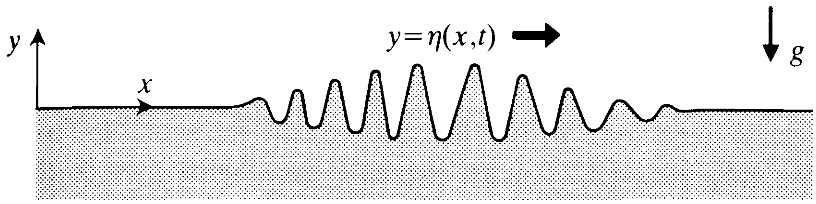


FIGURE 2: A deformation on the free surface of water in the form of a wave packet.

Kinematic condition at the free surface:

Fluid particles on the surface must remain on the surface.

The kinematic condition entails that $F(x, y, t) = y - \eta(x, t)$ **remains constant** (in fact, zero) for any particular particle on the free surface which means that

$$\frac{\mathrm{D}F}{\mathrm{D}t} = \frac{\partial F}{\partial t} + (\boldsymbol{u} \cdot \nabla)F = 0 \quad \text{on} \quad y = \eta(x, t), \tag{9}$$

and this is equivalent to

$$\frac{\partial \eta}{\partial t} + u\,\frac{\partial \eta}{\partial x} = v \quad \text{on} \quad y = \eta(x, t). \tag{10}$$

Pressure condition at the free surface:

The fluid is **inviscid** (by assumption), so the condition at the free surface is simply that the pressure there is equal to the atmospheric pressure p_0:

$$p = p_0 \quad \text{on} \quad y = \eta(x, t). \tag{11}$$

Bernoulli's equation for unsteady irrotational flow

If the flow is irrotational (so $\boldsymbol{u} = \nabla\phi$ and $\nabla \times \boldsymbol{u} = \boldsymbol{0}$), then, by integrating (over the space domain) the **Euler's momentum equation**:

$$\frac{\partial \nabla\phi}{\partial t} = -\nabla\left(\frac{p}{\varrho} + \frac{1}{2}\boldsymbol{u}^2 + \chi\right), \tag{12}$$

the **Bernoulli's equation** is obtained

$$\frac{\partial \phi}{\partial t} + \frac{p}{\varrho} + \frac{1}{2}\boldsymbol{u}^2 + \chi = G(t)\,. \tag{13}$$

Here, χ is the gravity potential (in the present context $\chi = g\,y$ where g is the gravity acceleration) and $G(t)$ is an arbitrary function of time alone (a constant of integration).

Now, by choosing $G(t)$ in a convenient manner, $G(t) = \frac{p_0}{\varrho}$, the **pressure condition** may be written as:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2\right) + g\,\eta = 0 \quad \text{on} \quad y = \eta(x, t). \tag{14}$$

Small-amplitude waves

The free surface displacement $\eta(x,t)$ and the fluid velocities u, v are small (in a sense to be made precise later).

- Linearization of the **kinematic condition**

$$v = \frac{\partial \eta}{\partial t} + \underbrace{u\,\frac{\partial \eta}{\partial x}}_{\text{small}} \quad \to \quad v(x,\eta,t) = \frac{\partial \eta}{\partial t}$$

$$\xrightarrow[\text{series}]{\text{Taylor}} \quad v(x,0,t) + \underbrace{\eta\,\frac{\partial v}{\partial y}(x,0,t) + \cdots}_{\text{small}} = \frac{\partial \eta}{\partial t} \tag{15}$$

$$\to \quad v(x,0,t) = \frac{\partial \eta}{\partial t} \quad \xrightarrow{v=\frac{\partial \phi}{\partial y}} \quad \boxed{\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \quad \text{on } y=0.}$$

- Linearization of the **pressure condition**

$$\frac{\partial \phi}{\partial t} + \underbrace{\frac{1}{2}\left(u^2+v^2\right)}_{\text{small}} + g\,\eta = 0 \quad \to \quad \boxed{\frac{\partial \phi}{\partial t} + g\,\eta = 0 \quad \text{on } y=0.} \tag{16}$$

A sinusoidal travelling wave solution

The **free surface** is of the form

$$\eta = A\,\cos(k\,x - \omega\,t)\,, \tag{17}$$

where A is the **amplitude** of the surface displacement, ω is the **circular frequency**, and k is the **circular wavenumber**.

- The corresponding **velocity potential** is

$$\phi = q(y)\,\sin(k\,x - \omega\,t)\,. \tag{18}$$

- It satisfies the Laplace's equation, $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$.
- Therefore, $q(y)$ must satisfy $q'' - k^2 q = 0$, the general solution of which is

$$q = C\,\exp(k\,y) + D\exp(-k\,y)\,. \tag{19}$$

- For *deep* water waves $D = 0$ (if $k > 0$ which may be assumed without loss of generality) in order that the velocity be bounded as $y \to -\infty$. Therefore, the velocity potential for *deep* water waves is

$$\phi = C\,\exp(k\,y)\,\sin(k\,x - \omega\,t)\,. \tag{20}$$

- Now, the (linearized) **free surface conditions** yield what follows:
  1. the **kinematic condition** ($\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$ on $y = 0$):

$$C\,k = A\,\omega \quad \rightarrow \quad \boxed{\phi = \frac{A\,\omega}{k}\exp(k\,y)\,\sin(k\,x - \omega\,t)\,,} \tag{21}$$

  2. the **pressure condition** ($\frac{\partial \phi}{\partial t} + g\,\eta = 0$ on $y = 0$):

$$-C\,\omega + g\,A = 0 \quad \rightarrow \quad \boxed{\omega^2 = g\,k\,.} \quad \textbf{(dispersion relation!)} \tag{22}$$

The **fluid velocity** components:

$$u = A\,\omega\,\exp(k\,y)\,\cos(k\,x - \omega\,t)\,, \qquad v = A\,\omega\,\exp(k\,y)\,\sin(k\,x - \omega\,t)\,. \tag{23}$$

Particle paths

Any particle departs only a **small amount** (X, Y) **from its mean position** (x, y). Therefore, its position as a function of time may be found by integrating $u = \frac{\mathrm{d}X}{\mathrm{d}t}$ and $v = \frac{\mathrm{d}Y}{\mathrm{d}t}$; whence:

$$X(t) = -A\,\exp(k\,y)\,\sin(k\,x - \omega\,t)\,, \quad Y(t) = A\,\exp(k\,y)\,\cos(k\,x - \omega\,t)\,. \tag{24}$$

Figure 3 presents particle paths for a wave on deep water. One may observe what follows:

- Particle **paths are circular**.
- The **radius** of the path circles, $A\,\exp(k\,y)$, **decrease** exponentially **with depth**. So do the **fluid velocities**.
- Virtually all the **energy** of a surface water wave is **contained within half a wavelength below the surface**.

Effects of finite depth

If the **fluid is bonded below** by a rigid plane $y = -h$, so that

$$v = \frac{\partial \phi}{\partial y} = 0 \quad \text{at } y = -h, \tag{25}$$

the **dispersion relation** and the **phase speed** are as follows:

$$\omega^2 = g\,k\,\tanh(k\,h)\,, \qquad c^2 = \frac{g}{k}\tanh(k\,h)\,. \tag{26}$$

Figure 4 shows the phase speed of waves in water of uniform depth h in function of the wavelength $\lambda = \frac{2\pi}{k}$. There are two limit cases:

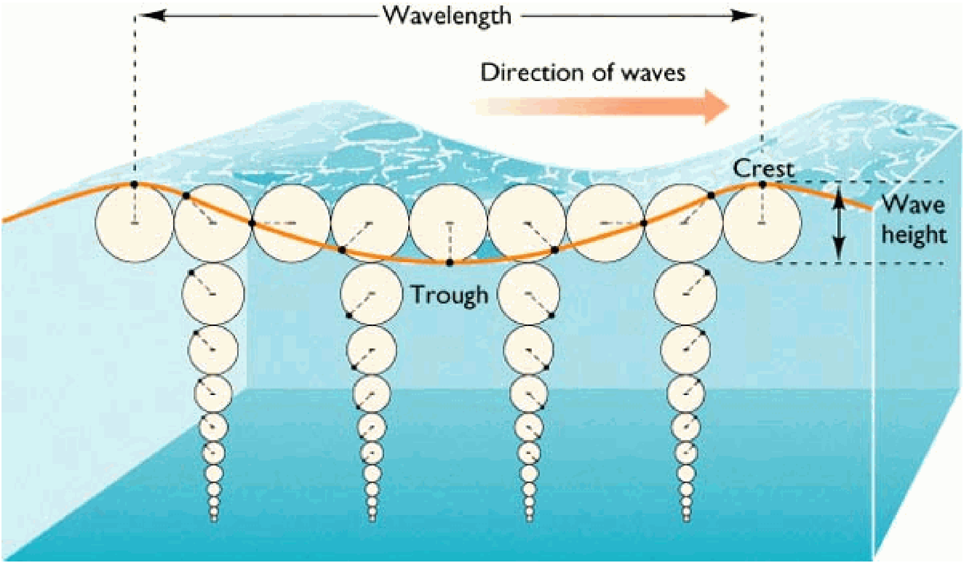


FIGURE 3: Deep water wave and circular particle paths (*Wikipedia*).

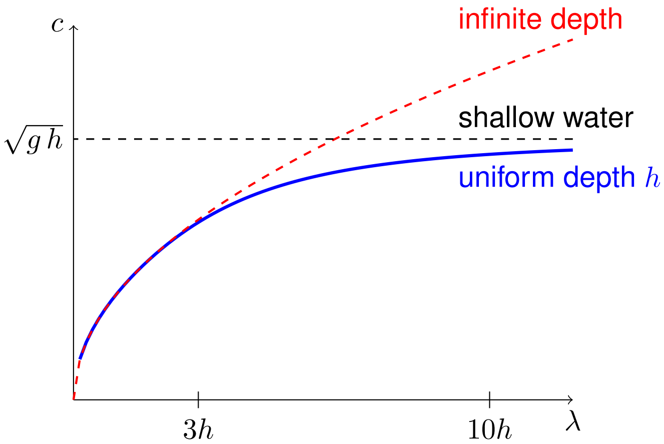


FIGURE 4: The phase speed of gravity waves in water of uniform depth h.

1. $h \gg \lambda$ (**infinite depth**): $k\,h = 2\pi\,\frac{h}{\lambda}$ is large and $\tanh(k\,h) \approx 1$, so $\boxed{c^2 = \frac{g}{k}}$. In practice, this is a good approximation if $h > \frac{1}{3}\lambda$.
2. $h \ll \lambda/2\pi$ (**shallow water**): $k\,h \ll 1$ and $\tanh(k\,h) \approx k\,h$, so $\boxed{c^2 = g\,h}$, which means that c is independent of k in this limit. Thus, the **gravity waves** in **shallow water** are **non-dispersive**.

2.2 Dispersion and the group velocity

Dispersion of waves

Dispersion of waves is the phenomenon that the **phase velocity of a wave depends on its frequency**.

There are generally two sources of dispersion:

1. the **material dispersion** comes from a frequency-dependent response of a material to waves

2. the **waveguide dispersion** occurs when the speed of a wave in a waveguide depends on its frequency for geometric reasons, independent of any frequency-dependence of the materials from which it is constructed.

Dispersion relation

The dispersion exists when the (angular) frequency is related to the wavenumber in a non-linear way:

$$\boxed{\omega = \omega(k)} = c(k)\,k\,, \qquad c = c(k) = \frac{\omega(k)}{k}\,. \tag{27}$$

If $\omega(k)$ is a **linear** function of k then c is **constant** and the medium is **non-dispersive**.

Dispersion relations for waves on water surface:

- deep water waves: $\omega = \sqrt{g\,k}$, $c = \sqrt{\frac{g}{k}}$.
- finite depth waves: $\omega = \sqrt{g\,k\tanh(k\,h)}$, $c = \sqrt{\frac{g}{k}\tanh(k\,h)}$.
- shallow water waves: $\omega = \sqrt{g\,h}\,k$, $c = \sqrt{g\,h}$ $\rightarrow$ non-dispersive!

Group and phase velocity

Two fundamental velocities of wave propagation, namely, the group velocity c_{g} and the phase velocity c, are defined as follows:

$$\boxed{c_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}}\,, \qquad c = \frac{\omega}{k}\,. \tag{28}$$

- In **dispersive systems** both velocities are different and frequency-dependent (i.e., wavenumber-dependent): $c_{\mathrm{g}} = c_{\mathrm{g}}(k)$ and $c = c(k)$.
- In **non-dispersive systems** they are equal and constant: $c_{\mathrm{g}} = c$.

Important properties of the group velocity:

1. At this velocity the isolated **wave packet** travels as *a whole*.

   Discussion for a wave packet (see Figure 5): for k in the neighbourhood of k_0

$$\omega(k) \approx \omega(k) + (k - k_0)\,c_{\mathrm{g}}\,, \quad \text{where} \;\; c_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}\Big|_{k=k_0}, \tag{29}$$

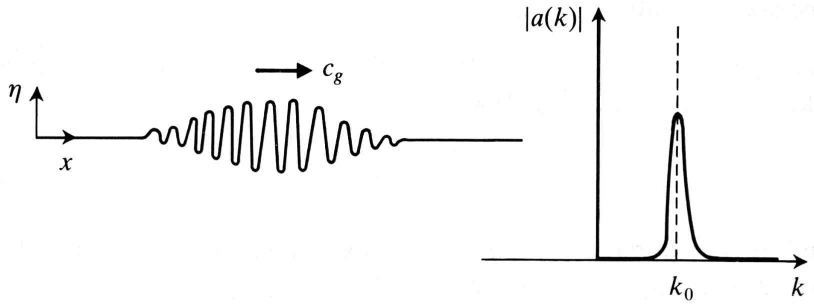


FIGURE 5: A wave packet and its spectrum.

and $\omega(k) = 0$ outside the neighbourhood; the Fourier integral equals

$$\eta(x,t) = \mathrm{Re}\left[\int_{-\infty}^{\infty} a(k)\, \exp\left(\mathrm{i}\,(k\,x - \omega\,t)\right) \mathrm{d}k\right] \quad \leftarrow \text{(for a general disturbance)}$$

$$\approx \mathrm{Re}\left[\overbrace{\exp\left(\mathrm{i}\left(k_0\,x - \omega(k_0)\,t\right)\right)}^{\text{a pure harmonic wave}} \int_{-\infty}^{\infty} \overbrace{a(k)\, \exp\left(\mathrm{i}\,(k - k_0)\,(x - c_\mathrm{g}\,t)\right) \mathrm{d}k}^{\text{a function of } (x - c_\mathrm{g}\,t)}\right]. \tag{30}$$

2. The **energy is transported** at the group velocity (by waves of a given wavelength).

3. One must travel at the group velocity to see the waves of the same wavelength.

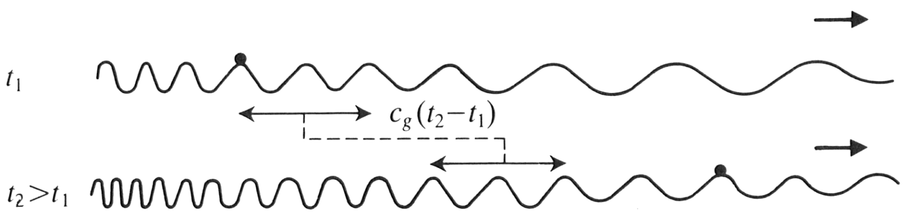


FIGURE 6: A train of waves.

A slowly varying **wavetrain** (see Figure 6) can be written as

$$\eta(x,t) = \mathrm{Re}\left[A(x,t)\, \exp\left(\mathrm{i}\,\theta(x,t)\right)\right], \tag{31}$$

where the **phase function** $\theta(x,t)$ describes the oscillatory aspect of the wave, while $A(x,t)$ describes the gradual modulation of its amplitude.

The *local* wavenumber and frequency are defined by

$$k = \frac{\partial\theta}{\partial x}, \quad \omega = -\frac{\partial\theta}{\partial t}. \tag{32}$$

For purely sinusoidal wave $\theta = k\,x - \omega\,t$, where k and ω are constants. In general, k and ω are functions of x and t. It follows immediately that

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \quad \longrightarrow \quad \frac{\partial k}{\partial t} + \frac{\mathrm{d}\omega}{\mathrm{d}k}\frac{\partial k}{\partial x} = \frac{\partial k}{\partial t} + c_\mathrm{g}(k)\,\frac{\partial k}{\partial x} = 0 \tag{33}$$

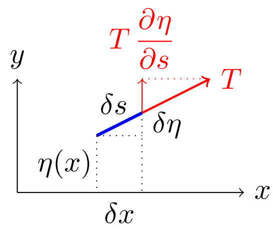
which means that $k(x,t)$ is constant for an observer moving with the velocity $c_\mathrm{g}(k)$.

2.3 Capillary waves

Surface tension

A **surface tension force** $T\ \left[\frac{\mathrm{N}}{\mathrm{m}}\right]$ is a force per unit length, directed tangentially to the surface, acting on a line drawn parallel to the wavecrests.

- The **vertical component** of surface tension force equals $T\,\dfrac{\partial\eta}{\partial s}$, where s denotes the distance along the surface.
- For **small wave amplitudes** $\delta s \approx \delta x$, and then $T\,\dfrac{\partial\eta}{\partial s} \approx T\,\dfrac{\partial\eta}{\partial x}$.



- A small portion of surface of length δx will experience surface tension at both ends, so the **net upward force** on it will be

$$T\,\frac{\partial\eta}{\partial x}\bigg|_{x+\delta x} - T\,\frac{\partial\eta}{\partial x}\bigg|_{x} = T\,\frac{\partial^2\eta}{\partial x^2}\,\delta x \tag{34}$$

- Therefore, an upward force **per unit area of surface** is $T\,\dfrac{\partial^2\eta}{\partial x^2}$.

Local equilibrium at the free surface

The net upward force per unit area of surface, $T\,\frac{\partial^2\eta}{\partial x^2}$, must be balanced by the difference between the atmospheric pressure p_0 and the pressure p in the fluid just below the surface:

$$p_0 - p = T\,\frac{\partial^2\eta}{\partial x^2} \quad \text{on} \quad y = \eta(x,t). \tag{35}$$

This **pressure condition** at the free surface takes into consideration the **effects of surface tension**. The kinematic condition remains the same: fluid particles cannot leave the surface.

Linearized free surface conditions (with surface tension effects)

For small amplitude waves:

$$\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t}\,, \qquad \frac{\partial\phi}{\partial t} + g\,\eta = \frac{T}{\varrho}\,\frac{\partial^2\eta}{\partial x^2} \qquad \text{on } y = 0. \tag{36}$$

Notice that the right-hand-side term of the pressure condition results from a surface tension.

A sinusoidal travelling wave solution $\eta = A\,\cos(k\,x - \omega\,t)$ leads now to a new **dispersion**

relation

$$\omega^2 = g\,k + \frac{T\,k^3}{\varrho}. \tag{37}$$

As a consequence, the **phase** and **group velocities** include now the surface tension effect:

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k} + \frac{T\,k}{\varrho}}\,, \qquad c_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{g + 3T\,k^2/\varrho}{2\sqrt{g\,k + T\,k^3/\varrho}}\,. \tag{38}$$

Surface tension importance parameter

The relative importance of surface tension and gravitational forces in a fluid is measured by the following parameter

$$\beta = \frac{T\,k^2}{\varrho\,g}\,. \tag{39}$$

(The so-called *Bond number* $= \frac{\varrho\,g\,L^2}{T}$; it equals $\frac{4\pi^2}{\beta}$ if $L = \lambda$.)

Now, the dispersion relation, as well as the phase and group velocities can be written as

$$\omega^2 = g\,k\,(1+\beta)\,, \qquad c = \sqrt{\frac{g}{k}(1+\beta)}\,, \qquad c_{\mathrm{g}} = \frac{g\,(1+3\beta)}{2\sqrt{g\,k\,(1+\beta)}}\,. \tag{40}$$

Depending on the parameter β, two extreme cases are distinguished:

1. $\beta \ll 1$: the effects of surface tension are negligible – the waves are **gravity waves** for which

$$\omega^2 = g\,k\,, \qquad c = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\,\lambda}{2\pi}}\,, \qquad c_{\mathrm{g}} = \frac{c}{2}\,. \tag{41}$$

2. $\beta \gg 1$: the waves are essentially **capillary waves** for which

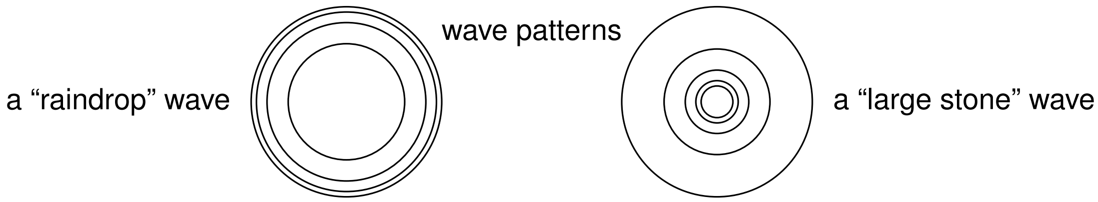
$$\omega^2 = g\,k\,\beta = \frac{T\,k^3}{\varrho}, \quad c = \sqrt{\frac{g}{k}\beta} = \sqrt{\frac{T\,k}{\varrho}} = \sqrt{\frac{2\pi\,T}{\varrho\,\lambda}}\,, \quad c_{\mathrm{g}} = \frac{g\,3\beta}{2\sqrt{g\,k\,\beta}} = \frac{3}{2}c\,. \tag{42}$$

CAPILLARY WAVES:

- **short waves** travel **faster**,
- the group velocity exceeds the phase velocity, $c_{\mathrm{g}} > c$,
- the **wavecrests move backward** through a wave packet as it moves along as a whole.

GRAVITY WAVES:

- **long waves** travel **faster**,
- the group velocity is less than the phase velocity, $c_{\mathrm{g}} < c$,
- the **wavecrests move faster** than a wave packet.



The **capillary effects** predominate when **raindrops** fall on a pond, and as short waves travel faster the **wavelength decreases with radius** at any particular time.

The **effects of gravity** predominate when a **large stone** is dropped into a pond (on account of the longer wavelengths involved), and as long waves travel faster the **wavelength increases with radius** at any particular time.

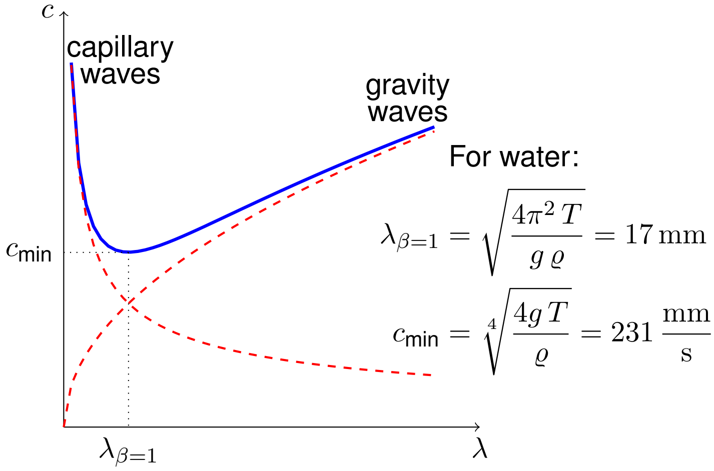


FIGURE 7: The phase speed for capillary-gravity waves.

For $\beta \approx 1$ both effects (the surface tension and gravity) are significant and the waves are **capillary-gravity waves**. Figure 7 presents the phase speed of such waves depending on the wavelength $\lambda = \frac{2\pi}{k}$. Notice that the speed reaches its minimum, $c = c_{\text{min}}$, for such λ that $\beta = 1$. For water at 20°C (when $T = 7.29 \times 10^{-4}\,\frac{\mathrm{N}}{\mathrm{m}}$ and $\varrho = 998\,\frac{\mathrm{kg}}{\mathrm{m}^3}$) this is when the wavelength is about 17 mm.

***Example*: Uniform flow past a submerged obstacle**

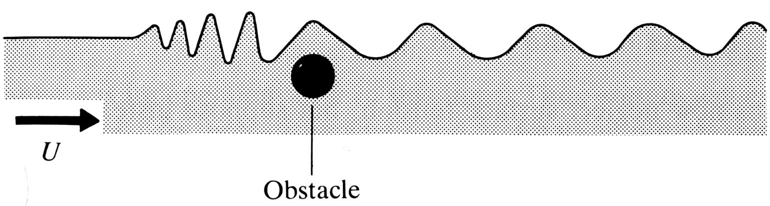


FIGURE 8: Stationary waves generated by uniform flow, speed U, past a submerged obstacle.

Figure 8 presents stationary waves generated by uniform flow past a submerged obstacle. Two cases are distinguished with respect to the flow speed U, namely:

1. $U < c_{\min}$ – there are no steady waves generated by the obstacle;
2. $U > c_{\min}$ – there are **two values** of λ ($\lambda_1 > \lambda_2$) for which $c = U$:

   λ_1 – **the larger value** represents a **gravity wave**:

   - the corresponding group velocity is less than c,
   - the **energy** of this relatively long-wavelength disturbance **is carried downstream** of the obstacle.

   λ_2 – **the smaller value** represents a **capillary wave**:

   - the corresponding group velocity is greater than c,
   - the **energy** of this relatively short-wavelength disturbance **is carried upstream** of the obstacle, where it is rather quickly dissipated by viscous effects, on account of the short wavelength (in fact, each wave-crest is at rest, but relative to still water it is travelling upstream with speed U).

2.4 Shallow-water finite-amplitude waves

***Assumptions*:**

- The **amplitudes of waves are finite**, that is, *not* (infinitesimally) small compared with the depth; therefore, the **linearized theory does not apply**.
- A typical value h_0 of depth $h(x,t)$ is much smaller than a typical horizontal length scale L of the wave (see Figure 9), that is: $\boxed{h_0 \ll L}$. This is the basis of the so-called **shallow-water approximation**.

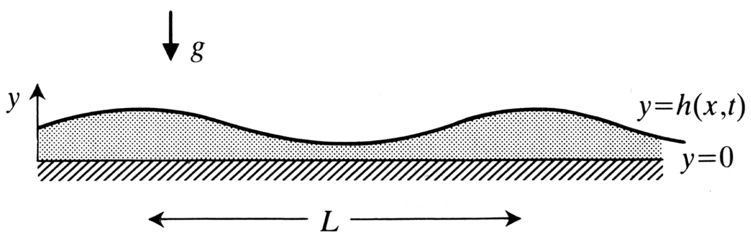


FIGURE 9: Finite-amplitude wave on shallow water.

▶ The full (nonlinear) 2-D equations are:

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\varrho}\frac{\partial p}{\partial x}\,, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{1}{\varrho}\frac{\partial p}{\partial y} - g\,, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,. \tag{43}$$

▶ In the shallow-water approximation (when $h_0 \ll L$) the vertical component of acceleration can be neglected in comparison with the gravitational acceleration:

$$\frac{\mathrm{D}v}{\mathrm{D}t} \ll g \quad \to \quad 0 = -\frac{1}{\varrho}\frac{\partial p}{\partial y} - g \quad \to \quad \frac{\partial p}{\partial y} = \varrho\, g\,. \tag{44}$$

Integrating and applying the condition $p = p_0$ at $y = h(x,t)$ gives

$$p(x,y,t) = p_0 - \varrho\, g\big[y - h(x,t)\big]\,. \tag{45}$$

This is used for the equation for the horizontal component of acceleration:

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -g\,\frac{\partial h}{\partial x} \quad \xrightarrow{\frac{\partial u}{\partial y}=0} \quad \boxed{\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = -g\,\frac{\partial h}{\partial x}} \tag{46}$$

where $u = u(x,t)$ and $h = h(x,t)$.

▶ A second equation linking u and h may be obtained as follows:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \quad \to \quad v(x,y,t) = -\frac{\partial u(x,t)}{\partial x}\,y + f(x,t) \quad \xrightarrow{v=0 \text{ at } y=0} \quad v = -\frac{\partial u}{\partial x}\,y\,, \tag{47}$$

and using the **kinematic condition at the free surface** – fluid particles on the surface must remain on it, so the vertical component of velocity v equals the rate of change of the depth h when moving with the horizontal velocity u:

$$v = \frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} \quad \text{at } y = h(x,t) \quad \to \quad \boxed{\frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} + h\,\frac{\partial u}{\partial x} = 0}\,. \tag{48}$$

Shallow-water equations

Nonlinear equations for the horizontal component of velocity $u = u(x,t)$ and the depth $h = h(x,t)$ of finite-amplitude waves on shallow water:

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + g\,\frac{\partial h}{\partial x} = 0\,, \qquad \frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} + h\,\frac{\partial u}{\partial x} = 0\,. \tag{49}$$

(The vertical component of velocity is $v(x,y,t) = -\frac{\partial u}{\partial x}\,y$.)
On introducing the new variable $\boxed{c(x,t) = \sqrt{g\,h}}$ and then adding and subtracting the two equations the form suited to treatment by the *method of characteristics* is obtained

$$\left[\frac{\partial}{\partial t} + (u+c)\,\frac{\partial}{\partial x}\right](u+2c) = 0\,, \qquad \left[\frac{\partial}{\partial t} + (u-c)\,\frac{\partial}{\partial x}\right](u-2c) = 0\,. \tag{50}$$

Let $x = x(s)$, $t = t(s)$ be a **characteristic curve** defined parametrically (s is the parameter) in the x-t plane and starting at some point (x_0, t_0). In fact, two such (families of) characteristic curves are defined such that:

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1\,, \qquad \frac{\mathrm{d}x}{\mathrm{d}s} = u \pm c\,. \tag{51}$$

This (with $+$) is used for the first and (with $-$) for the second equation:

$$\left[\frac{\mathrm{d}t}{\mathrm{d}s}\,\frac{\partial}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}s}\,\frac{\partial}{\partial x}\right](u \pm 2c) = 0 \quad \xrightarrow{\text{the chain rule}} \quad \boxed{\frac{\mathrm{d}}{\mathrm{d}s}\,(u \pm 2c) = 0}\,. \tag{52}$$

General property: $u \pm 2c$ is constant along 'positive'/'negative' characteristic curves defined by $\frac{\mathrm{d}x}{\mathrm{d}t} = u \pm c$.

Within the framework of the theory of finite-amplitude waves on shallow water the following problems can be solved:

- the dam-break flow,
- the formation of a bore,
- the hydraulic jump.

3 Sound waves

3.1 Introduction

Sound waves propagate due to the **compressibility** of a medium ($\nabla \cdot \boldsymbol{u} \neq 0$). Depending on frequency one can distinguish:

- **infrasound waves** – below 20 Hz,
- **acoustic waves** – from 20 Hz to 20 kHz,
- **ultrasound waves** – above 20 kHz.

Acoustics deals with vibrations and waves in compressible continua in the **audible frequency range**, that is, from 20 Hz (16 Hz) to 20 000 Hz.

Types of waves in compressible continua:

- an **inviscid compressible fluid** – (only) longitudinal waves,
- an infinite **isotropic solid** – longitudinal and shear waves,
- an **anisotropic solid** – wave propagation is more complex.

3.2 Acoustic wave equation

***Assumptions*:**

- Gravitational forces can be neglected so that the equilibrium (undisturbed-state) pressure and density take on uniform values, p_0 and ϱ_0, throughout the fluid.
- Dissipative effects, that is viscosity and heat conduction, are neglected.
- The medium (fluid) is homogeneous, isotropic, and perfectly elastic.

Small-amplitudes assumption

Particle velocity is small, and there are only very small perturbations (fluctuations) to the equilibrium pressure and density:

$$\boldsymbol{u} - \text{small}\,, \qquad p = p_0 + \tilde{p} \quad (\tilde{p} - \text{small})\,, \qquad \varrho = \varrho_0 + \tilde{\varrho} \quad (\tilde{\varrho} - \text{small})\,. \tag{53}$$

The pressure fluctuations field $\tilde{p}$ is called the **acoustic pressure**.

Momentum equation (Euler's equation):

$$\varrho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p \quad \xrightarrow{\text{linearization}} \quad \varrho_0\,\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p\,. \tag{54}$$

Notice that $\nabla p = \nabla(p_0 + \tilde{p}) = \nabla\tilde{p}$.

Continuity equation:

$$\frac{\partial \varrho}{\partial t} + \nabla\cdot(\varrho\,\boldsymbol{u}) = 0 \quad \xrightarrow{\text{linearization}} \quad \frac{\partial \tilde{\varrho}}{\partial t} + \varrho_0\,\nabla\cdot\boldsymbol{u} = 0\,. \tag{55}$$

Using divergence operation for the linearized momentum equation and time-differentiation for the linearized continuity equation yields:

$$\frac{\partial^2 \tilde{\varrho}}{\partial t^2} - \triangle p = 0\,. \tag{56}$$

Constitutive relation:

$$p = p(\tilde{\varrho}) \quad \rightarrow \quad \frac{\partial p}{\partial t} = \frac{\partial p}{\partial \tilde{\varrho}}\,\frac{\partial \tilde{\varrho}}{\partial t} \quad \rightarrow \quad \frac{\partial^2 \tilde{\varrho}}{\partial t^2} = \frac{1}{c_0^2}\,\frac{\partial^2 p}{\partial t^2} \quad \text{where } c_0^2 = \frac{\partial p}{\partial \tilde{\varrho}}\,. \tag{57}$$

Wave equation for the pressure field

$$\boxed{\frac{1}{c_0^2}\,\frac{\partial^2 p}{\partial t^2} - \triangle p = 0} \quad \text{where} \quad c_0 = \sqrt{\frac{\partial p}{\partial \tilde{\varrho}}} \tag{58}$$

is the **acoustic wave velocity** (or the **speed of sound**). Notice that the acoustic pressure $\tilde{p}$ can be used here instead of p. Moreover, the wave equation for the density-fluctuation field $\tilde{\varrho}$ (or for the compression field $\tilde{\varrho}/\varrho_0$), for the velocity potential ϕ, and for the velocity field $\boldsymbol{u}$ can be derived analogously.

3.3 The speed of sound

Inviscid isotropic elastic liquid. The pressure in an inviscid liquid depends on the volume dilatation $\operatorname{tr}\boldsymbol{\varepsilon}$:

$$p = -K\,\operatorname{tr}\boldsymbol{\varepsilon}\,, \tag{59}$$

where K is the bulk modulus. Now,

$$\frac{\partial p}{\partial t} = -K\,\operatorname{tr}\frac{\partial \boldsymbol{\varepsilon}}{\partial t} = -K\,\nabla\cdot\boldsymbol{u} \quad \xrightarrow[\text{Lin. Cont. Eq.}]{\nabla\cdot\boldsymbol{u} = -\frac{1}{\varrho_0}\frac{\partial\tilde{\varrho}}{\partial t}} \quad \frac{\partial p}{\partial t} = \frac{K}{\varrho_0}\,\frac{\partial \tilde{\varrho}}{\partial t} \tag{60}$$

which means that the speed of sound $c_0 = \sqrt{\partial p / \partial \tilde{\varrho}}$ is given by the well-known formula:

$$\boxed{c_0 = \sqrt{\frac{K}{\varrho_0}}}. \tag{61}$$

Perfect gas. The determination of speed of sound in a perfect gas is complicated and requires the use of thermodynamic considerations. The final result is

$$c_0 = \sqrt{\gamma \frac{p_0}{\varrho_0}} = \sqrt{\gamma R T_0}\,, \tag{62}$$

where γ denotes the ratio of specific heats ($\gamma = 1.4$ for air), R is the universal gas constant, and T_0 is the (isothermal) temperature.

▶ For air at $20°$C and normal atmospheric pressure: $c_0 = 343\,\frac{\mathrm{m}}{\mathrm{s}}$.

3.4 Sub- and supersonic flow

A steady, unseparated, **compressible flow** past a thin airfoil may be written in the from

$$u = U + \frac{\partial \phi}{\partial x}\,, \qquad v = \frac{\partial \phi}{\partial y}\,, \tag{63}$$

where the **velocity potential** ϕ for the small disturbance to the uniform flow U satisfies

$$(1 - M^2)\,\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0\,, \qquad \text{where} \qquad \boxed{M = \frac{U}{c_0}} \tag{64}$$

is the **Mach number** defined as the ratio of the speed of free stream to the speed of sound.

▶ If $M^2 \ll 1$ that gives the Laplace equation which is the result that arises for **incompressible theory** (i.e., using $\nabla \cdot \boldsymbol{u} = 0$).

▶ Otherwise, three cases can be distinguished:

1. $M < 1$ – the **subsonic flow** (see Figure 10):
   - there is **some disturbance to the oncoming flow** at all distances from the wing (even though it is very small when the distance is large);
   - the **drag is zero** (inviscid theory) and the lift $= \frac{\text{lift}_{\text{incompressible}}}{\sqrt{1-M^2}}$.
2. $M > 1$ – the **supersonic flow** (see Figure 11):
   - there is **no disturbance to the oncoming stream** except between the **Mach lines** extending from the ends of the airfoil and making the angle $\alpha = \arcsin\left(\frac{1}{M}\right)$ with the uniform stream;
   - the **drag is not zero** – it arises because of the sound wave energy which the wing radiates to infinity between the Mach lines.

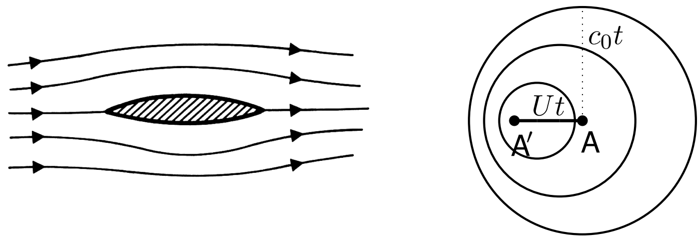


FIGURE 10: (*Left*:) Subsonic flow past a thin wing at zero incidence. (*Right*:) Acoustic radiation by a body moving subsonically ($M = 0.6$).

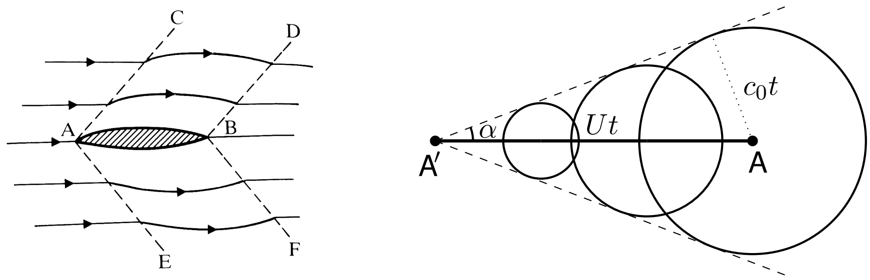


FIGURE 11: (*Left*:) Supersonic flow past a thin wing at zero incidence. (*Right*:) Acoustic radiation by a body moving supersonically ($M = 2.8$).

3. $M \approx 1$ – the **sound barrier**:

- sub- and supersonic theory is not valid;
- nonetheless, it indicates that the wing is subject to a destructive effect of exceptionally **large aerodynamic forces**.