

Fundamentals of Fluid Dynamics: Waves in Fluids

Introductory Course on Multiphysics Modelling

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(after: D.J. ACHESON's "*Elementary Fluid Dynamics*")

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Outline

1 Introduction

- The notion of wave
- Basic wave phenomena
- Mathematical description of a traveling wave

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- Surface waves on deep water
- Dispersion and the group velocity
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The notion of wave

What is a wave?

A **wave** is the **transport of a disturbance** (or energy, or piece of information) in space not associated with motion of the medium occupying this space as a whole. (Except that electromagnetic waves require no medium !!!)

- The transport is at **finite speed**.
- The shape or form of the **disturbance** is **arbitrary**.
- The disturbance moves with respect to the medium.

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- 1 longitudinal waves** – the disturbance moves parallel to the direction of propagation. *Examples:* sound waves, compressional elastic waves (P-waves in geophysics);
- 2 transverse waves** – the disturbance moves perpendicular to the direction of propagation. *Examples:* waves on a string or membrane, shear waves (S-waves in geophysics), water waves, electromagnetic waves.

Basic wave phenomena

reflection – change of wave direction from hitting a reflective surface,

refraction – change of wave direction from entering a new medium,

diffraction – wave circular spreading from entering a small hole (of the wavelength-comparable size), or wave bending around small obstacles,

interference – superposition of two waves that come into contact with each other,

dispersion – wave splitting up by frequency,

rectilinear propagation – the movement of light wave in a straight line.

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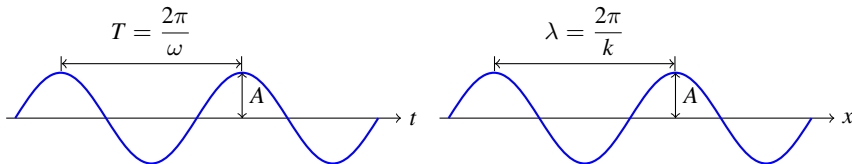
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Standing wave

A **standing wave**, also known as a **stationary wave**, is a wave that remains in a constant position. This phenomenon can occur:

- when the medium is moving in the opposite direction to the wave,
- (in a stationary medium:) as a result of interference between two waves travelling in opposite directions.

Mathematical description of a harmonic wave



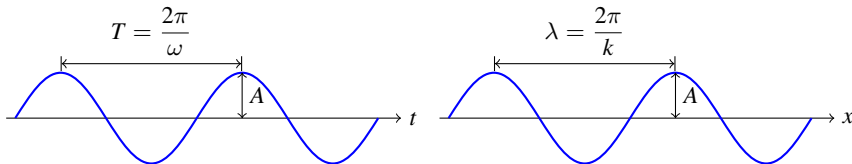
Traveling waves

Simple wave or **traveling wave**, sometimes also called *progressive wave*, is a disturbance that varies both with time t and distance x in the following way:

$$\begin{aligned}
 u(x, t) &= A(x, t) \cos(kx - \omega t + \theta_0) \\
 &= A(x, t) \sin\left(kx - \omega t + \underbrace{\theta_0 \pm \frac{\pi}{2}}_{\tilde{\theta}_0}\right)
 \end{aligned}$$

where A is the **amplitude**, ω and k denote the **angular frequency** and **wavenumber**, and θ_0 (or $\tilde{\theta}_0$) is the initial **phase**.

Mathematical description of a harmonic wave

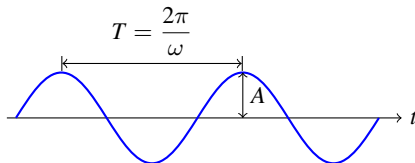


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 \end{aligned}$$

- **Amplitude** A [e.g. m, Pa, V/m] – a measure of the maximum disturbance in the medium during one wave cycle (the maximum distance from the highest point of the crest to the equilibrium).
- **Phase** $\theta = kx - \omega t + \theta_0$ [rad], where θ_0 is the *initial* phase (shift), often ambiguously, called the phase.

Mathematical description of a harmonic wave



- **Period** T [s] – the time for one complete cycle for an oscillation of a wave.
- **Frequency** f [Hz] – the number of periods per unit time.

Frequency and angular frequency

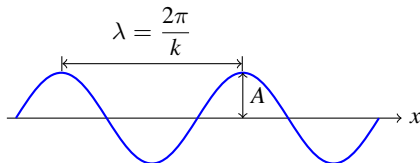
The **frequency** f [Hz] represents the number of periods per unit time

$$f = \frac{1}{T}.$$

The **angular frequency** ω [Hz] represents the frequency in terms of radians per second. It is related to the frequency by

$$\omega = \frac{2\pi}{T} = 2\pi f.$$

Mathematical description of a harmonic wave



- **Wavelength** λ [m] – the distance between two sequential crests (or troughs).

Wavenumber and angular wavenumber

The **wavenumber** is the spatial analogue of frequency, that is, it is the measurement of the number of repeating units of a propagating wave (the number of times a wave has the same phase) per unit of space.

*Application of a Fourier transformation on data as a function of time yields a **frequency spectrum**; application on data as a function of position yields a **wavenumber spectrum**.*

The **angular wavenumber** k $\left[\frac{1}{\text{m}}\right]$, often misleadingly abbreviated as “wave-number”, is defined as

$$k = \frac{2\pi}{\lambda}.$$

Mathematical description of a harmonic wave

There are two velocities that are associated with waves:

1 Phase velocity – the rate at which the wave propagates:

$$c = \frac{\omega}{k} = \lambda f.$$

Mathematical description of a harmonic wave

There are two velocities that are associated with waves:

- 1 **Phase velocity** – the rate at which the wave propagates:

$$c = \frac{\omega}{k} = \lambda f.$$

- 2 **Group velocity** – the velocity at which variations in the shape of the wave's amplitude (known as the *modulation* or *envelope* of the wave) propagate through space:

$$c_g = \frac{d\omega}{dk}.$$

This is (in most cases) the signal velocity of the waveform, that is, the **rate at which information or energy is transmitted** by the wave. However, if the wave is travelling through an absorptive medium, this does not always hold.

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Surface waves on deep water

- Consider **two-dimensional** water waves: $\mathbf{u} = [u(x, y, t), v(x, y, t), 0]$.
- Suppose that the flow is **irrotational**: $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$.
- Therefore, there exists a **velocity potential** $\phi(x, y, t)$ so that

$$u = \frac{\partial \phi}{\partial x} , \quad v = \frac{\partial \phi}{\partial y} .$$

- The fluid is **incompressible**, so by the virtue of the incompressibility condition, $\nabla \cdot \mathbf{u} = 0$, the velocity potential ϕ will satisfy **Laplace's equation**

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 .$$

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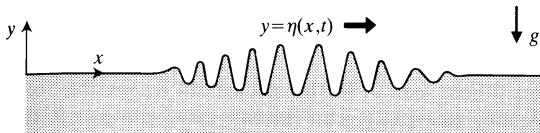
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Free surface

The fluid motion arises from a deformation of the water surface. The equation of this free surface is denoted by $y = \eta(x, t)$.



Surface waves on deep water

Kinematic condition at the free surface

Kinematic condition at the free surface:

Fluid particles on the surface must remain on the surface.

The kinematic condition entails that $F(x, y, t) = y - \eta(x, t)$ **remains constant** (in fact, zero) for any particular particle on the free surface which means that

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + (\mathbf{u} \cdot \nabla)F = 0 \quad \text{on} \quad y = \eta(x, t),$$

and this is equivalent to

$$\frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} = v \quad \text{on} \quad y = \eta(x, t).$$

Surface waves on deep water

Pressure condition at the free surface

Pressure condition at the free surface:

The fluid is **inviscid** (by assumption), so the condition at the free surface is simply that the pressure there is equal to the atmospheric pressure p_0 :

$$p = p_0 \quad \text{on} \quad y = \eta(x, t).$$

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Bernoulli's equation for unsteady irrotational flow

If the flow is irrotational (so $\mathbf{u} = \nabla\phi$ and $\nabla \times \mathbf{u} = \mathbf{0}$), then, by integrating (over the space domain) the **Euler's momentum equation**:

$$\frac{\partial \nabla \phi}{\partial t} = -\nabla \left(\frac{p}{\rho} + \frac{1}{2} \mathbf{u}^2 + \chi \right),$$

the **Bernoulli's equation** is obtained

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2} \mathbf{u}^2 + \chi = G(t).$$

Here, χ is the gravity potential (in the present context $\chi = g y$ where g is the gravity acceleration) and $G(t)$ is an arbitrary function of time alone (a constant of integration).

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$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2} \mathbf{u}^2 + g y = G(t).$$

Here, $G(t)$ is an arbitrary function of time alone (a constant of integration).

Now, by choosing $G(t)$ in a convenient manner, $G(t) = \frac{p_0}{\rho}$, the **pressure condition** may be written as:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (u^2 + v^2) + g \eta = 0 \quad \text{on} \quad y = \eta(x, t).$$

Surface waves on deep water

Small amplitude waves: the linearized surface conditions

Small-amplitude waves

The free surface displacement $\eta(x, t)$ and the fluid velocities u, v are small.

■ Linearization of the **kinematic condition**

$$\begin{aligned}
 v &= \frac{\partial \eta}{\partial t} + \underbrace{u \frac{\partial \eta}{\partial x}}_{\text{small}} \rightarrow v(x, \eta, t) = \frac{\partial \eta}{\partial t} \\
 &\xrightarrow{\text{Taylor series}} v(x, 0, t) + \underbrace{\eta \frac{\partial v}{\partial y}(x, 0, t) + \dots}_{\text{small}} = \frac{\partial \eta}{\partial t} \\
 &\rightarrow v(x, 0, t) = \frac{\partial \eta}{\partial t} \xrightarrow{v = \frac{\partial \phi}{\partial y}} \boxed{\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \quad \text{on } y = 0.}
 \end{aligned}$$

■ Linearization of the **pressure condition**

$$\frac{\partial \phi}{\partial t} + \underbrace{\frac{1}{2}(u^2 + v^2)}_{\text{small}} + g\eta = 0 \rightarrow \boxed{\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{on } y = 0.}$$

Surface waves on deep water

Dispersion relation and travelling wave solution

A sinusoidal travelling wave solution

The **free surface** is of the form

$$\eta = A \cos(kx - \omega t),$$

where A is the **amplitude** of the surface displacement, ω is the **circular frequency**, and k is the **circular wavenumber**.

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- The corresponding **velocity potential** is

$$\phi = q(y) \sin(kx - \omega t).$$

- It satisfies the Laplace's equation, $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$.
- Therefore, $q(y)$ must satisfy $q'' - k^2 q = 0$, the general solution of which is

$$q = C \exp(ky) + D \exp(-ky).$$

- For *deep water waves* $D = 0$ (if $k > 0$ which may be assumed without loss of generality) in order that the velocity be bounded as $y \rightarrow -\infty$.

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- Now, the (linearized) **free surface conditions** yield what follows:

- 1 the **kinematic condition** ($\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$ on $y = 0$):

$$Ck = A\omega \quad \rightarrow \quad \boxed{\phi = \frac{A\omega}{k} \exp(ky) \sin(kx - \omega t)},$$

- 2 the **pressure condition** ($\frac{\partial \phi}{\partial t} + g\eta = 0$ on $y = 0$):

$$-C\omega + gA = 0 \quad \rightarrow \quad \boxed{\omega^2 = gk} \quad (\text{dispersion relation!})$$

Surface waves on deep water

Particle paths

The **fluid velocity** components:

$$u = A \omega \exp(k y) \cos(k x - \omega t), \quad v = A \omega \exp(k y) \sin(k x - \omega t).$$

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Any particle departs only a **small amount** (X, Y) **from its mean position** (x, y) . Therefore, its position as a function of time may be found by integrating $u = \frac{dX}{dt}$ and $v = \frac{dY}{dt}$; whence:

$$X(t) = -A \exp(k y) \sin(k x - \omega t), \quad Y(t) = A \exp(k y) \cos(k x - \omega t).$$

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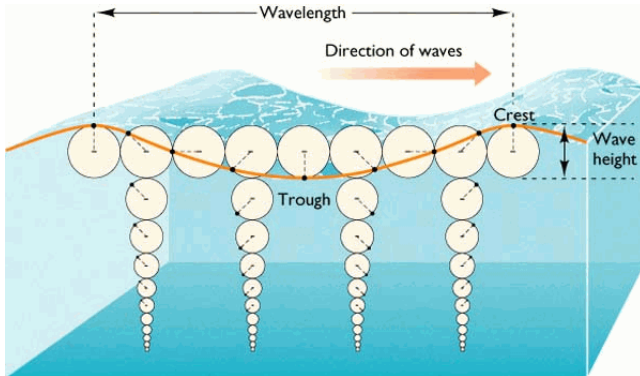
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- Particle **paths are circular**.
- The **radius** of the path circles, $A \exp(ky)$, **decrease** exponentially **with depth**. So do the **fluid velocities**.
- Virtually all the **energy** of a surface water wave is **contained within half a wavelength below the surface**.

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Surface waves on deep water

Effects of finite depth

Effects of finite depth

If the **fluid is bonded below** by a rigid plane $y = -h$, so that

$$v = \frac{\partial \phi}{\partial y} = 0 \quad \text{at } y = -h,$$

the **dispersion relation** and the **phase speed** are as follows:

$$\omega^2 = g k \tanh(k h), \quad c^2 = \frac{g}{k} \tanh(k h).$$

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There are two limit cases:

- 1** $h \gg \lambda$ (**infinite depth**): $k h = 2\pi \frac{h}{\lambda}$ is large and $\tanh(k h) \approx 1$, so

$c^2 = \frac{g}{k}$. In practice, this is a good approximation if $h > \frac{1}{3}\lambda$.

- 2** $h \ll \lambda/2\pi$ (**shallow water**): $k h \ll 1$ and $\tanh(k h) \approx k h$, so

$c^2 = g h$, which means that c is independent of k in this limit.

Thus, the **gravity waves** in **shallow water** are **non-dispersive**.

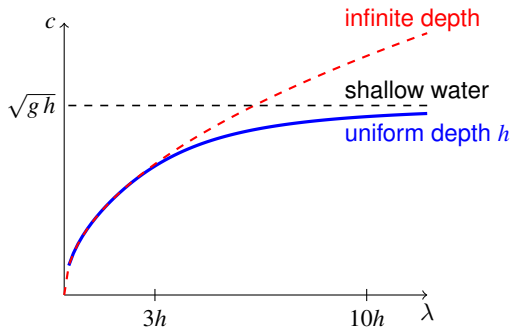
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Dispersion and the group velocity

Dispersion of waves

Dispersion of waves is the phenomenon that the **phase velocity of a wave depends on its frequency**.

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There are generally two sources of dispersion:

- 1 the **material dispersion** comes from a frequency-dependent response of a material to waves
- 2 the **waveguide dispersion** occurs when the speed of a wave in a waveguide depends on its frequency for geometric reasons, independent of any frequency-dependence of the materials from which it is constructed.

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Dispersion relation

$$\boxed{\omega = \omega(k)} = c(k) k, \quad c = c(k) = \frac{\omega(k)}{k}.$$

If $\omega(k)$ is a **linear** function of k then c is **constant** and the medium is **non-dispersive**.

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- ▶ deep water waves: $\omega = \sqrt{g k}$, $c = \sqrt{\frac{g}{k}}$.
- ▶ finite depth waves: $\omega = \sqrt{g k \tanh(k h)}$, $c = \sqrt{\frac{g}{k} \tanh(k h)}$.
- ▶ shallow water waves: $\omega = \sqrt{g h} k$, $c = \sqrt{g h} \rightarrow$ **non-dispersive!**

Dispersion and the group velocity

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Group and phase velocity

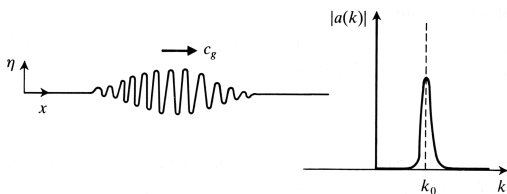
$$\boxed{c_g = \frac{d\omega}{dk}}, \quad c = \frac{\omega}{k}.$$

- In **dispersive systems** both velocities are different and frequency-dependent (i.e., wavenumber-dependent): $c_g = c_g(k)$ and $c = c(k)$.
- In **non-dispersive systems** they are equal and constant: $c_g = c$.

Dispersion and the group velocity

Important properties of the group velocity:

- 1 At this velocity the isolated **wave packet** travels as *a whole*.



Discussion for a wave packet: for k in the neighbourhood of k_0

$$\omega(k) \approx \omega(k_0) + (k - k_0) c_g, \quad \text{where } c_g = \left. \frac{d\omega}{dk} \right|_{k=k_0},$$

and $\omega(k) = 0$ outside the neighbourhood; the Fourier integral equals

$$\eta(x, t) = \operatorname{Re} \left[\int_{-\infty}^{\infty} a(k) \exp(i(kx - \omega t)) dk \right] \quad \leftarrow \text{(for a general disturbance)}$$

$$\approx \operatorname{Re} \left[\overbrace{\exp(i(k_0 x - \omega(k_0) t))}^{\text{a pure harmonic wave}} \int_{-\infty}^{\infty} \overbrace{a(k) \exp(i(k - k_0)(x - c_g t))}^{\text{a function of } (x - c_g t)} dk \right].$$

Dispersion and the group velocity

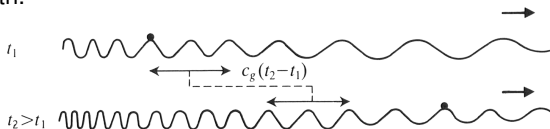
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Dispersion and the group velocity

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- 3 One must travel at the group velocity to see the waves of the same wavelength.



A slowly varying **wavetrain** can be written as

$$\eta(x, t) = \text{Re} \left[A(x, t) \exp \left(i \theta(x, t) \right) \right],$$

where the **phase function** $\theta(x, t)$ describes the oscillatory aspect of the wave, while $A(x, t)$ describes the gradual modulation of its amplitude.

The *local* wavenumber and frequency are defined by

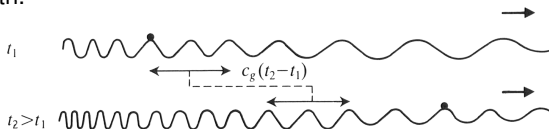
$$k = \frac{\partial \theta}{\partial x}, \quad \omega = - \frac{\partial \theta}{\partial t}.$$

For purely sinusoidal wave $\theta = kx - \omega t$, where k and ω are constants.

Dispersion and the group velocity

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- 3 One must travel at the group velocity to see the waves of the same wavelength.



The *local* wavenumber and frequency are defined by

$$k = \frac{\partial \theta}{\partial x}, \quad \omega = -\frac{\partial \theta}{\partial t}.$$

For purely sinusoidal wave $\theta = kx - \omega t$, where k and ω are constants. In general, k and ω are functions of x and t . It follows immediately that

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \quad \longrightarrow \quad \frac{\partial k}{\partial t} + \frac{d\omega}{dk} \frac{\partial k}{\partial x} = \frac{\partial k}{\partial t} + c_g(k) \frac{\partial k}{\partial x} = 0$$

which means that $k(x, t)$ is constant for an observer moving with the velocity $c_g(k)$.

Capillary waves

Surface tension

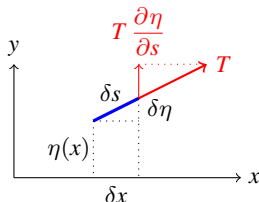
A **surface tension force** $T \left[\frac{\text{N}}{\text{m}} \right]$ is a force per unit length, directed tangentially to the surface, acting on a line drawn parallel to the wavecrests.

Capillary waves

Surface tension

A **surface tension force** $T \left[\frac{\text{N}}{\text{m}} \right]$ is a force per unit length, directed tangentially to the surface, acting on a line drawn parallel to the wavecrests.

- The **vertical component** of surface tension force equals $T \frac{\partial \eta}{\partial s}$, where s denotes the distance along the surface.
- For **small wave amplitudes** $\delta s \approx \delta x$, and then $T \frac{\partial \eta}{\partial s} \approx T \frac{\partial \eta}{\partial x}$.

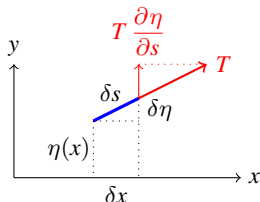


Capillary waves

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- For **small wave amplitudes** $\delta s \approx \delta x$, and then $T \frac{\partial \eta}{\partial s} \approx T \frac{\partial \eta}{\partial x}$.
- A small portion of surface of length δx will experience surface tension at both ends, so the **net upward force** on it will be



$$T \frac{\partial \eta}{\partial x} \Big|_{x+\delta x} - T \frac{\partial \eta}{\partial x} \Big|_x = T \frac{\partial^2 \eta}{\partial x^2} \delta x$$

- Therefore, an upward force **per unit area of surface** is $T \frac{\partial^2 \eta}{\partial x^2}$.

Capillary waves

Local equilibrium at the free surface

The net upward force per unit area of surface, $T \frac{\partial^2 \eta}{\partial x^2}$, must be balanced by the difference between the atmospheric pressure p_0 and the pressure p in the fluid just below the surface:

$$p_0 - p = T \frac{\partial^2 \eta}{\partial x^2} \quad \text{on} \quad y = \eta(x, t).$$

Capillary waves

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This **pressure condition** at the free surface takes into consideration the **effects of surface tension**. The kinematic condition remains the same: fluid particles cannot leave the surface.

Linearized free surface conditions (with surface tension effects)

For small amplitude waves:

$$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}, \quad \frac{\partial \phi}{\partial t} + g \eta = \frac{T}{\rho} \frac{\partial^2 \eta}{\partial x^2} \quad \text{on } y = 0.$$

Capillary waves

Linearized free surface conditions (with surface tension effects)

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A sinusoidal travelling wave solution $\eta = A \cos(kx - \omega t)$ leads now to a new **dispersion relation**

$$\omega^2 = gk + \frac{Tk^3}{\rho}.$$

As a consequence, the **phase** and **group velocities** include now the surface tension effect:

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k} + \frac{Tk}{\rho}}, \quad c_g = \frac{d\omega}{dk} = \frac{g + 3Tk^2/\rho}{2\sqrt{gk + Tk^3/\rho}}.$$

Capillary waves

Surface tension importance parameter

The relative importance of surface tension and gravitational forces in a fluid is measured by the following parameter

$$\beta = \frac{T k^2}{\rho g}.$$

(The so-called *Bond number* $= \frac{\rho g L^2}{T}$; it equals $\frac{4\pi^2}{\beta}$ if $L = \lambda$.)

Now, the dispersion relation, as well as the phase and group velocities can be written as

$$\omega^2 = g k (1 + \beta), \quad c = \sqrt{\frac{g}{k} (1 + \beta)}, \quad c_g = \frac{g (1 + 3\beta)}{2\sqrt{g k (1 + \beta)}}.$$

Capillary waves

Surface tension importance parameter

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Depending on the parameter β , two extreme cases are distinguished:

- 1 $\beta \ll 1$: the effects of surface tension are negligible – the waves are **gravity waves** for which

$$\omega^2 = g k, \quad c = \sqrt{\frac{g}{k}} = \sqrt{\frac{g \lambda}{2\pi}}, \quad c_g = \frac{c}{2}.$$

Capillary waves

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- 2 $\beta \gg 1$: the waves are essentially **capillary waves** for which

$$\omega^2 = g k \beta = \frac{T k^3}{\rho}, \quad c = \sqrt{\frac{g}{k} \beta} = \sqrt{\frac{T k}{\rho}} = \sqrt{\frac{2\pi T}{\rho \lambda}}, \quad c_g = \frac{g 3\beta}{2\sqrt{g k \beta}} = \frac{3}{2} c.$$

Capillary waves vs. gravity waves

CAPILLARY WAVES:

- **short waves** travel **faster**,

GRAVITY WAVES:

- **long waves** travel **faster**,

Capillary waves vs. gravity waves

CAPILLARY WAVES:

- **short waves** travel **faster**,
- the group velocity exceeds the phase velocity, $c_g > c$,

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Capillary waves vs. gravity waves

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- the **crests move backward** through a wave packet as it moves along as a whole.

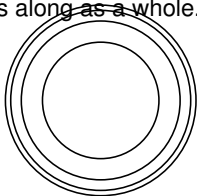
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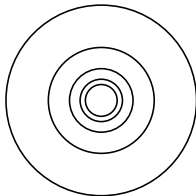


wave patterns

The **capillary effects** predominate when **raindrops** fall on a pond, and as short waves travel faster the **wavelength decreases with radius** at any particular time.

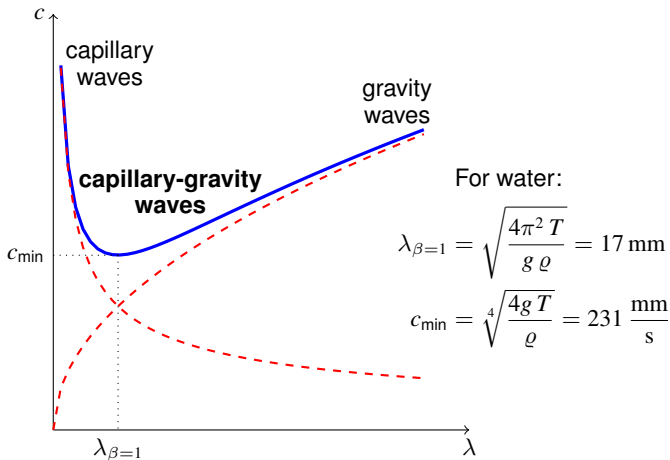
GRAVITY WAVES:

- **long waves** travel **faster**,
- the group velocity is less than the phase velocity, $c_g < c$,
- the **wavecrests move faster** than a wave packet.



The **effects of gravity** predominate when a **large stone** is dropped into a pond, and as long waves travel faster the **wavelength increases with radius** at any particular time.

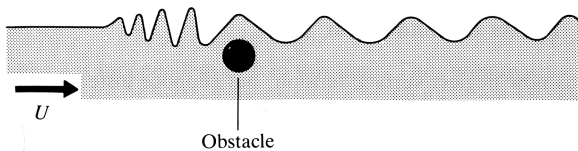
Capillary-gravity waves



For $\beta \approx 1$ both effects (the surface tension and gravity) are significant and the waves are **capillary-gravity waves**.

Capillary waves vs. gravity waves

Example: Uniform flow past a submerged obstacle

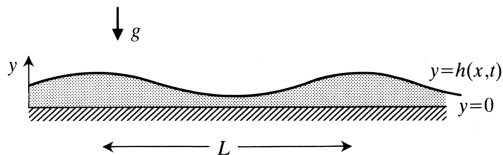


- 1 $U < c_{\min}$ – there are no steady waves generated by the obstacle;
- 2 $U > c_{\min}$ – there are **two values** of λ ($\lambda_1 > \lambda_2$) for which $c = U$:
 - λ_1 – **the larger value** represents a **gravity wave**:
 - the corresponding group velocity is less than c ,
 - the **energy** of this relatively long-wavelength disturbance **is carried downstream** of the obstacle.
 - λ_2 – **the smaller value** represents a **capillary wave**:
 - the corresponding group velocity is greater than c ,
 - the **energy** of this relatively short-wavelength disturbance **is carried upstream** of the obstacle, where it is rather quickly dissipated by viscous effects, on account of the short wavelength (in fact, each wave-crest is at rest, but relative to still water it is travelling upstream with speed U).

Shallow-water finite-amplitude waves

Assumptions:

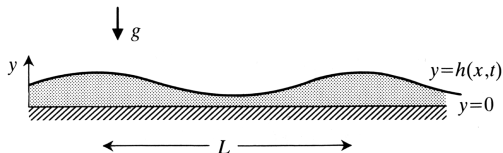
- The **amplitudes of waves are finite**, that is, *not* (infinitesimally) small compared with the depth; therefore, the **linearized theory does not apply**.
- A typical value h_0 of depth $h(x, t)$ is much smaller than a typical horizontal length scale L of the wave, that is: $h_0 \ll L$. This is the basis of the so-called **shallow-water approximation**.



- The full (nonlinear) 2-D equations are:

$$\frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad \frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - g, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Shallow-water finite-amplitude waves



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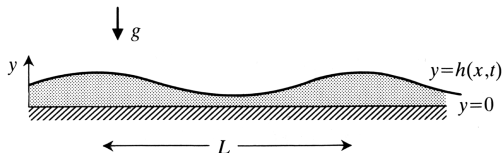
► In the shallow-water approximation (when $h_0 \ll L$) the vertical component of acceleration can be neglected in comparison with the gravitational acceleration:

$$\frac{Dv}{Dt} \ll g \quad \rightarrow \quad 0 = -\frac{1}{\varrho} \frac{\partial p}{\partial y} - g \quad \rightarrow \quad \frac{\partial p}{\partial y} = \varrho g.$$

Integrating and applying the condition $p = p_0$ at $y = h(x, t)$ gives

$$p(x, y, t) = p_0 - \varrho g [y - h(x, t)].$$

Shallow-water finite-amplitude waves



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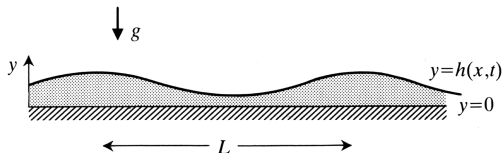
$$p(x, y, t) = p_0 - \varrho g [y - h(x, t)].$$

This is used for the equation for the horizontal component of acceleration:

$$\frac{Du}{Dt} = -g \frac{\partial h}{\partial x} \xrightarrow{\frac{\partial u}{\partial y} = 0} \boxed{\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial h}{\partial x}}$$

where $u = u(x, t)$ and $h = h(x, t)$.

Shallow-water finite-amplitude waves



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► A second equation linking u and h may be obtained as follows:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \rightarrow v(x, y, t) = -\frac{\partial u(x, t)}{\partial x} y + f(x, t) \quad \xrightarrow{v=0 \text{ at } y=0} \quad v = -\frac{\partial u}{\partial x} y,$$

and using the **kinematic condition at the free surface** – fluid particles on the surface must remain on it, so the vertical component of velocity v equals the rate of change of the depth h when moving with the horizontal velocity u :

$$v = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} \quad \text{at } y = h(x, t) \rightarrow \boxed{\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0}.$$

Shallow-water finite-amplitude waves

Shallow-water equations

Nonlinear equations for the horizontal component of velocity $u = u(x, t)$ and the depth $h = h(x, t)$ of finite-amplitude waves on shallow water:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = 0, \quad \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0.$$

(The vertical component of velocity is $v(x, y, t) = -\frac{\partial u}{\partial x} y$.)

On introducing the new variable $c(x, t) = \sqrt{gh}$ and then adding and subtracting the two equations the form suited to treatment by the *method of characteristics* is obtained

$$\left[\frac{\partial}{\partial t} + (u + c) \frac{\partial}{\partial x} \right] (u + 2c) = 0, \quad \left[\frac{\partial}{\partial t} + (u - c) \frac{\partial}{\partial x} \right] (u - 2c) = 0.$$

Shallow-water finite-amplitude waves

Shallow-water equations

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Let $x = x(s)$, $t = t(s)$ be a **characteristic curve** defined parametrically (s is the parameter) in the x - t plane and starting at some point (x_0, t_0) . In fact, two such (families of) characteristic curves are defined such that:

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = u \pm c.$$

This (with $+$) is used for the first and (with $-$) for the second equation:

$$\left[\frac{dt}{ds} \frac{\partial}{\partial t} + \frac{dx}{ds} \frac{\partial}{\partial x} \right] (u \pm 2c) = 0 \quad \xrightarrow{\text{the chain rule}} \quad \frac{d}{ds} (u \pm 2c) = 0.$$

Shallow-water finite-amplitude waves

Shallow-water equations

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Let $x = x(s) = x(t)$, $t = t(s) = s$ be a **characteristic curve** such that:

$$\frac{dx}{dt} = u \pm c.$$

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General property: $u \pm 2c$ is constant along ‘positive’/‘negative’ characteristic curves defined by $\frac{dx}{dt} = u \pm c$.

Shallow-water finite-amplitude waves

Shallow-water equations

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = 0, \quad \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0.$$

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Within the framework of the theory of finite-amplitude waves on shallow water the following problems can be solved:

- the dam-break flow,
- the formation of a bore,
- the hydraulic jump.

Outline

1 Introduction

- The notion of wave
- Basic wave phenomena
- Mathematical description of a traveling wave

2 Water waves

- Surface waves on deep water
- Dispersion and the group velocity
- Capillary waves
- Shallow-water finite-amplitude waves

3 Sound waves

- Introduction
- Acoustic wave equation
- The speed of sound
- Sub- and supersonic flow

Sound waves: introduction

Sound waves propagate due to the **compressibility** of a medium ($\nabla \cdot \mathbf{u} \neq 0$). Depending on frequency one can distinguish:

- **infrasound waves** – below 20 Hz,
- **acoustic waves** – from 20 Hz to 20 kHz,
- **ultrasound waves** – above 20 kHz.

Acoustics deals with vibrations and waves in compressible continua in the **audible frequency range**, that is, from 20 Hz (16 Hz) to 20 000 Hz.

Types of waves in compressible continua:

- an **inviscid compressible fluid** – (only) longitudinal waves,
- an infinite **isotropic solid** – longitudinal and shear waves,
- an **anisotropic solid** – wave propagation is more complex.

Acoustic wave equation

Assumptions:

- Gravitational forces can be neglected so that the equilibrium (undisturbed-state) pressure and density take on uniform values, p_0 and ϱ_0 , throughout the fluid.
- Dissipative effects, that is viscosity and heat conduction, are neglected.
- The medium (fluid) is homogeneous, isotropic, and perfectly elastic.

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Small-amplitudes assumption

Particle velocity is small, and there are only very small perturbations (fluctuations) to the equilibrium pressure and density:

$$\mathbf{u} - \text{small}, \quad p = p_0 + \tilde{p} \quad (\tilde{p} - \text{small}), \quad \varrho = \varrho_0 + \tilde{\varrho} \quad (\tilde{\varrho} - \text{small}).$$

The pressure fluctuations field \tilde{p} is called the **acoustic pressure**.

Acoustic wave equation

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The pressure fluctuations field \tilde{p} is called the **acoustic pressure**.

Momentum equation (Euler's equation):

$$\varrho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p \quad \xrightarrow{\text{linearization}} \quad \varrho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p.$$

Notice that $\nabla p = \nabla(p_0 + \tilde{p}) = \nabla \tilde{p}$.

Continuity equation:

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{u}) = 0 \quad \xrightarrow{\text{linearization}} \quad \frac{\partial \tilde{\varrho}}{\partial t} + \varrho_0 \nabla \cdot \mathbf{u} = 0.$$

Acoustic wave equation

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Using divergence operation for the linearized momentum equation and time-differentiation for the linearized continuity equation yields:

$$\frac{\partial^2 \tilde{\varrho}}{\partial t^2} - \Delta p = 0.$$

Acoustic wave equation

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Constitutive relation:

$$p = p(\tilde{\varrho}) \quad \rightarrow \quad \frac{\partial p}{\partial t} = \frac{\partial p}{\partial \tilde{\varrho}} \frac{\partial \tilde{\varrho}}{\partial t} \quad \rightarrow \quad \frac{\partial^2 \tilde{\varrho}}{\partial t^2} = \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} \quad \text{where } c_0^2 = \frac{\partial p}{\partial \tilde{\varrho}}.$$

Acoustic wave equation

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Wave equation for the pressure field

$$\boxed{\frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} - \Delta p = 0} \quad \text{where } c_0 = \sqrt{\frac{\partial p}{\partial \tilde{\varrho}}}$$

is the **acoustic wave velocity** (or the **speed of sound**). Notice that the acoustic pressure \tilde{p} can be used here instead of p . Moreover, the wave equation for the density-fluctuation field $\tilde{\varrho}$ (or for the compression field $\tilde{\varrho}/\varrho_0$), for the velocity potential ϕ , and for the velocity field \mathbf{u} can be derived analogously.

The speed of sound

Inviscid isotropic elastic liquid. The pressure in an inviscid liquid depends on the volume dilatation $\text{tr } \varepsilon$:

$$p = -K \text{tr } \varepsilon ,$$

where K is the bulk modulus. Now,

$$\frac{\partial p}{\partial t} = -K \text{tr } \frac{\partial \varepsilon}{\partial t} = -K \nabla \cdot \mathbf{u} \quad \xrightarrow[\text{Lin. Cont. Eq.}]{\nabla \cdot \mathbf{u} = -\frac{1}{\varrho_0} \frac{\partial \tilde{\varrho}}{\partial t}} \quad \frac{\partial p}{\partial t} = \frac{K}{\varrho_0} \frac{\partial \tilde{\varrho}}{\partial t}$$

which means that the speed of sound $c_0 = \sqrt{\partial p / \partial \tilde{\varrho}}$ is given by the well-known formula:

$$c_0 = \sqrt{\frac{K}{\varrho_0}} .$$

The speed of sound

Inviscid isotropic elastic liquid. The speed of sound is given by the well-known formula:

$$c_0 = \sqrt{\frac{K}{\rho_0}}.$$

Perfect gas. The determination of speed of sound in a perfect gas is complicated and requires the use of thermodynamic considerations. The final result is

$$c_0 = \sqrt{\gamma \frac{p_0}{\rho_0}} = \sqrt{\gamma R T_0},$$

where γ denotes the ratio of specific heats ($\gamma = 1.4$ for air), R is the universal gas constant, and T_0 is the (isothermal) temperature.

► For air at 20°C and normal atmospheric pressure: $c_0 = 343 \frac{\text{m}}{\text{s}}$.

Sub- and supersonic flow

A steady, unseparated, **compressible flow** past a thin airfoil may be written in the form

$$u = U + \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y},$$

where the **velocity potential** ϕ for the small disturbance to the uniform flow U satisfies

$$(1 - M^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \text{where} \quad M = \frac{U}{c_0}$$

is the **Mach number** defined as the ratio of the speed of free stream to the speed of sound.

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► Otherwise, three cases can be distinguished:

1 $M < 1$ – the **subsonic flow**

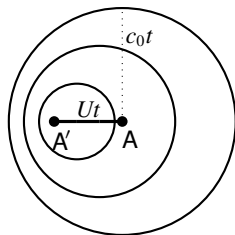
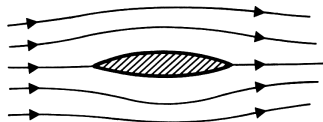
2 $M > 1$ – the **supersonic flow**

3 $M \approx 1$ – the **sound barrier**

Sub- and supersonic flow

1 $M < 1$ – the **subsonic flow**:

- there is **some disturbance to the oncoming flow** at all distances from the wing (even though it is very small when the distance is large);
- the **drag is zero** (inviscid theory) and the lift = $\frac{\text{lift}_{\text{incompressible}}}{\sqrt{1-M^2}}$.

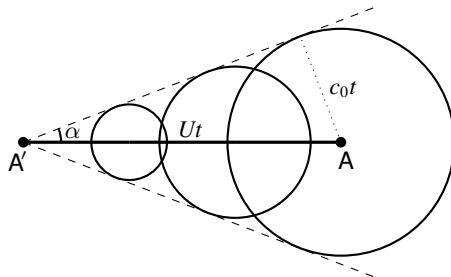
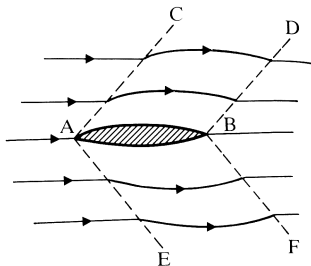


Sub- and supersonic flow

1 $M < 1$ – the **subsonic flow**

2 $M > 1$ – the **supersonic flow**:

- there is **no disturbance to the oncoming stream** except between the **Mach lines** extending from the ends of the airfoil and making the angle $\alpha = \arcsin\left(\frac{1}{M}\right)$ with the uniform stream;
- the **drag is not zero** – it arises because of the sound wave energy which the wing radiates to infinity between the Mach lines.



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- the **drag is not zero** – it arises because of the sound wave energy which the wing radiates to infinity between the Mach lines.

3 $M \approx 1$ – the **sound barrier**:

- sub- and supersonic theory is not valid;
- nonetheless, it indicates that the wing is subject to a destructive effect of exceptionally **large aerodynamic forces**.