Introduction to Finite Element Method

Introductory Course on Multiphysics Modelling

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Outline

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Motivation and general concepts

The **Finite Element Method (FEM)** is

- **generally speaking:** a powerful computational technique for the solution of differential and integral equations that arise in various fields of engineering and applied sciences;
- **mathematically:** a generalization of the classical variational (Ritz) and weighted-residual (Galerkin, least-squares, etc.) methods.

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Motivation

Most of the real problems:

- are defined on domains that are geometrically complex,
- may have different boundary conditions on different portions of the boundary.

Therefore, it is usually impossible (or difficult):

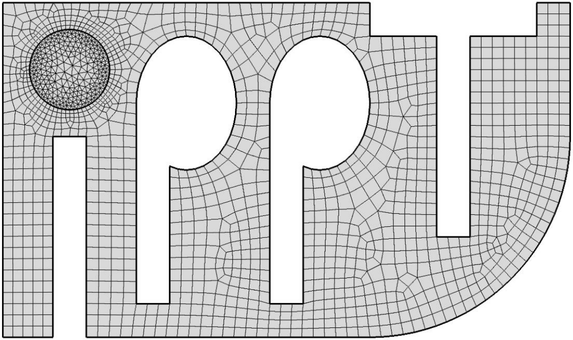
1. to find a solution analytically (so one must resort to approximate methods),
2. to generate approximation functions required in the traditional variational methods.

An answer to these problems is a **finite-element approach**.

Motivation and general concepts

Main concept of FEM

A problem domain can be viewed as an assemblage of simple geometric shapes, called **finite elements**, for which it is possible to systematically generate the approximation functions.



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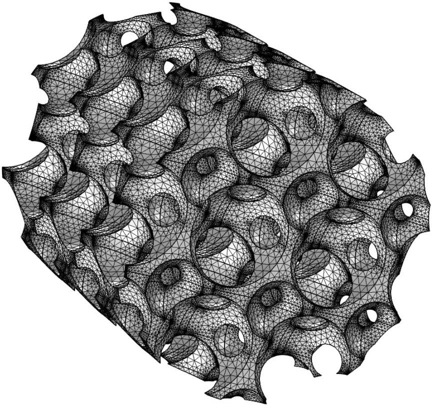

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6. **Solution of the system of algebraic equations** to find (approximate) values in the global degrees of freedom.
7. **Post-computation** of solution and quantities of interest.

Outline

Model problem

(O)DE: $$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha(x)\,\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)+\gamma(x)\,u(x)=f(x)\quad\text{for } x\in(a,b)$$

- $\alpha(x)$, $\gamma(x)$, $f(x)$ are the known data of the problem: the first two quantities result from the *material properties* and *geometry* of the problem whereas the third one depends on *source* or *loads*,
- $u(x)$ is the solution to be determined; it is also called **dependent variable** of the problem (with x being the **independent variable**).

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The domain of this 1D problem is an interval (a,b); the points $x = a$ and $x = b$ are the boundary points where **boundary conditions** are imposed, for examples, as follows

$$\textbf{BCs:} \quad \begin{cases} \left(q(a)\,n_x(a) = \right) - \alpha(a)\,\dfrac{\mathrm{d}u}{\mathrm{d}x}(a) = \hat{q}\,, & \text{(Neumann b.c.)} \\ u(b) = \hat{u}\,. & \text{(Dirichlet b.c.)} \end{cases}$$

- $\hat{q}$ and $\hat{u}$ are the given boundary values,
- n_x is the component of the outward unit vector normal to the boundary. In the 1D case there is only one component and: $n_x(a) = -1$, $n_x(b) = +1$.

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Moreover:

- $q(x)\equiv\alpha(x)\,\dfrac{\mathrm{d}u(x)}{\mathrm{d}x}$ is the so-called **secondary variable** specified on the boundary by the **Neumann boundary condition** also known as the **second kind** or **natural** boundary condition,
- $u(x)$ is the **primary variable** specified on the boundary by the **Dirichlet boundary condition** also known as the **first kind** or **essential** boundary condition.

Examples of different physics problems

u (primary var.)	α (material data)	f (source, load)	q (secondary var.)
Heat transfer			
temperature	thermal conductance	heat generation	heat
Flow through porous medium			
fluid-head	permeability	infiltration	source
Flow through pipes			
pressure	pipe resistance	0	source
Flow of viscous fluids			
velocity	viscosity	pressure gradient	shear stress
Elastic cables			
displacement	tension	transversal force	point force
Elastic bars			
displacement	axial stiffness	axial force	point force
Torsion of bars			
angle of twist	shear stiffness	0	torque
Electrostatics			
electric potential	dielectric constant	charge density	electric flux

Boundary Value Problem and the strong form

Let:

- $\Omega = (a, b)$ be an open set (an open interval in case of 1D problems);
- Γ be the boundary of Ω, that is, $\Gamma = \{a, b\}$;
- $\Gamma = \Gamma_q \cup \Gamma_u$ where, e.g., $\Gamma_q = \{a\}$ and $\Gamma_u = \{b\}$ are disjoint parts of the boundary (i.e., $\Gamma_q \cap \Gamma_u = \emptyset$) relating to the Neumann and Dirichlet boundary conditions, respectively;
- (the data of the problem): $f : \Omega \to \Re$, $\alpha : \Omega \to \Re$, $\gamma : \Omega \to \Re$;
- (the values prescribed on the boundary): $\hat{q} : \Gamma_q \to \Re$, $\hat{u} : \Gamma_u \to \Re$.

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Boundary Value Problem (BVP)

Find $u = ?$ satisfying

differential eq.:	$-\left(\alpha\, u'\right)' + \gamma\, u = f$	in $\Omega = (a,b)\,,$
Neumann b.c.:	$\alpha\, u'\, n_x = \hat{q}$	on $\Gamma_q = \{a\}\,,$
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Definition (Strong form)

The classical **strong form** of a boundary-value problem consists of:

- the **differential equation** of the problem,
- the **Neumann boundary conditions**, i.e., the natural conditions imposed on the secondary dependent variable (which involves the first derivative of the dependent variable).

The Dirichlet (essential) boundary conditions must be satisfied *a priori*.

The weak form

Derivation of weak form and the equivalence to strong form

Derivation of the equivalent weak form consists of the three steps presented below.

1. Write the ***weighted-residual*** **statement** for the equation.
2. Trade differentiation from u to δu using **integration by parts**.
3. **Use the Neumann boundary condition** ($\alpha\, u'\, n_x = \hat{q}$ on Γ_q) and the property of test function ($\delta u = 0$ on Γ_u) for the boundary term.

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$$\int_a^b \Big[-\left(\alpha\, u'\right)' + \gamma\, u - f \Big] \delta u \,\mathrm{d}x = 0\,.$$

   Here:

   - δu (the weighting function) belongs to the space of **test functions**,
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$$\Big[-\alpha\, u'\, \delta u \Big]_a^b + \int_a^b \Big[\alpha\, u'\, \delta u' + \gamma\, u\, \delta u - f\, \delta u \Big] \,\mathrm{d}x = 0\,.$$

Here, the boundary term may be written as

$$\begin{aligned}\Big[-\alpha\, u'\, \delta u \Big]_a^b &= \Big[-\alpha\, u'\, \delta u \Big]_{x=b} - \Big[-\alpha\, u'\, \delta u \Big]_{x=a} \\ &= \Big[-\alpha\, u'\, n_x\, \delta u \Big]_{x=b} + \Big[-\alpha\, u'\, n_x\, \delta u \Big]_{x=a} = \Big[-\alpha\, u'\, n_x\, \delta u \Big]_{x=\{a,b\}}\,.\end{aligned}$$

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The integration by parts *weakens the differentiability requirement* for the trial functions u (i.e., for the solution).

3. **Use the Neumann boundary condition** ($\alpha\,u'\,n_x = \hat{q}$ on Γ_q) and the property of test function ($\delta u = 0$ on Γ_u) for the boundary term

$$\Big[-\alpha\,u'\,n_x\,\delta u \Big]_{x=\{a,b\}} = \Big[-\underbrace{\alpha\,u'\,n_x}_{\hat{q}}\,\delta u \Big]_{x=a} + \Big[-\alpha\,u'\,n_x\,\underbrace{\delta u}_{0} \Big]_{x=b} = \Big[-\hat{q}\,\delta u \Big]_{x=a}\,.$$

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In this way, the **weak (variational) form** is obtained.

Weak form

$$\Big[-\,\hat{q}\,\delta u\Big]_{x=a} + \int\limits_a^b \Big[\alpha\, u'\,\delta u' + \gamma\, u\,\delta u - f\,\delta u\Big]\,\mathrm{d}x = 0\,.$$

The weak form is *mathematically equivalent* to the strong one: if u is a solution to the strong (local, differential) formulation of a BVP, it also satisfies the corresponding weak (global, integral) formulation for any δu (admissible, i.e., sufficiently smooth and $\delta u = 0$ on Γ_u).

The weak form

Additional requirements and remarks

- The essential boundary conditions must be explicitly satisfied by the trial functions: $u = \hat{u}$ on Γ_u. (In case of displacement formulations of many mechanical and structural engineering problems this is called **kinematic admissibility requirement**.)

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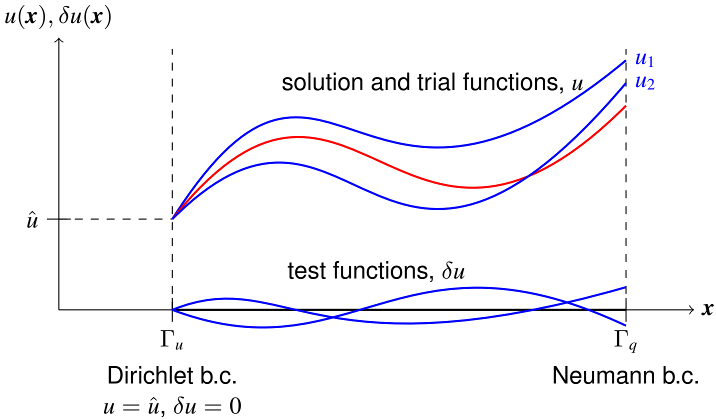
Remarks:

- The strong form can be derived from the corresponding weak formulation if more demanding assumptions are taken for the smoothness of trial functions (i.e., one-order higher differentiability).
- In variational methods, any test function is a variation defined as the difference between any two trial functions. Since any trial function satisfy the essential boundary conditions, the requirement that $\delta u = 0$ on Γ_u follows immediately.

The weak form

Test and trial functions



u_1, u_2 – arbitrary trial functions

$$\delta u = u_1 - u_2 \quad \text{and} \quad \left.\begin{array}{l} u_1 = \hat{u} \quad \text{on } \Gamma_u \\ u_2 = \hat{u} \quad \text{on } \Gamma_u \end{array}\right\} \quad \rightarrow \quad \delta u = 0 \quad \text{on } \Gamma_u$$

Associated variational problem

- $\mathcal{U}$, $\mathcal{W}$ are functional spaces. The first one is called the **space of solution** (or trial functions), the other one is the **space of test functions** (or weighting functions),
- $\mathcal{A}$ is a **bilinear form** defined on $\mathcal{U} \times \mathcal{W}$,
- $\mathcal{F}$ is a **linear form** defined on $\mathcal{W}$,
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The weak form is equivalent to a variational problem!

Weak form vs. variational problem

Weak formulation: Find $u \in \mathcal{U}$ so that $\mathcal{A}(u, \delta u) = \mathcal{F}(\delta u) \;\; \forall\, \delta u \in \mathcal{W}$.

Variational problem: Find $u \in \mathcal{U}$ which minimizes $\mathcal{P}(u)$.

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Example (for the model problem)

$$\mathcal{A}(u, \delta u) = \int_a^b \Big[\alpha\, u'\, \delta u' + \gamma\, u\, \delta u\Big]\, \mathrm{d}x\,, \qquad \mathcal{F}(\delta u) = \int_a^b f\, \delta u\, \mathrm{d}x + \Big[\hat{q}\, \delta u\Big]_{x=a}.$$

Associated variational problem

and the principle of the minimum total potential energy

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The weak form (or the variational problem) is the statement of the **principle of the minimum total potential energy**:

$$\boxed{\delta \mathcal{P}(u) = 0}, \qquad \delta \mathcal{P}(u) = \mathcal{A}(u, \delta u) - \mathcal{F}(\delta u)$$

- δ is now the **variational symbol**,
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- $\mathcal{P}(u)$ is the **potential energy** defined by the following **quadratic functional**

$$\mathcal{P}(u) = \frac{1}{2}\mathcal{A}(u, u) - \mathcal{F}(u)\,.$$

This definition holds only when the **bilinear form is *symmetric*** since:

$$\frac{1}{2}\,\delta\mathcal{A}(u, u) = \frac{1}{2}\Big(\underbrace{\mathcal{A}(\delta u, u)}_{\mathcal{A}(u,\delta u)} + \mathcal{A}(u, \delta u)\Big) = \mathcal{A}(u, \delta u)\,, \qquad \delta\mathcal{F}(u) = \mathcal{F}(\delta u)\,.$$

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$$\mathcal{P}(u) = \frac{1}{2}\mathcal{A}(u,u) - \mathcal{F}(u) = \int_a^b \left[\frac{\alpha}{2}\left(u'\right)^2 + \frac{\gamma}{2}\,u^2 - f\,u\right]\mathrm{d}x - \Big[\hat{q}\,u\Big]_{x=a},$$

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Outline

Galerkin method

Discrete (approximated) problem

If the problem is *well-posed* one can try to find an **approximated solution** u_h by solving the so-called **discrete problem** which is an approximation of the corresponding variational problem.

Discrete (approximated) problem

Find $u_h \in \mathcal{U}_h$ so that

$$\mathcal{A}_h(u_h, \delta u_h) = \mathcal{F}_h(\delta u_h) \quad \forall\, \delta u_h \in \mathcal{W}_h\,.$$

Here:

- $\mathcal{U}_h$ is a finite-dimension space of functions called **approximation space** whereas u_h is the **approximate solution** (i.e., *approximate* to the *original* problem).
- δu_h are **discrete test functions** from the **discrete test space** $\mathcal{W}_h$. In the Galerkin method $\boxed{\mathcal{W}_h = \mathcal{U}_h}$. (In general, $\mathcal{W}_h \neq \mathcal{U}_h$.)
- $\mathcal{A}_h$ is an approximation of the bilinear form $\mathcal{A}$.
- $\mathcal{F}_h$ is an approximation of the linear form $\mathcal{F}$.

Galerkin method

The interpolation and system of algebraic equations

In the Galerkin method ($\mathcal{W} = \mathcal{U}$) *the same* **shape functions**, $\phi_i(x)$, are used to ***interpolate*** the approximate solution as well as the (discrete) test functions:

$$u_h(x) = \sum_{j=1}^{N} \theta_j \, \phi_j(x) \,, \qquad \delta u_h(x) = \sum_{i=1}^{N} \delta\theta_i \, \phi_i(x) \,.$$

Here, θ_i are called the **degrees of freedom**.

Galerkin method

The interpolation and system of algebraic equations

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Using this **interpolation for the approximated problem** leads to a system of algebraic equations (as described below).

- The left-hand and right-hand sides of the problem equation yield:

$$\mathcal{A}_h(u_h, \delta u_h) = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathcal{A}_h(\phi_j, \phi_i) \; \theta_j \, \delta\theta_i = \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} \; \theta_j \, \delta\theta_i \,,$$

$$\mathcal{F}_h(\delta u_h) = \sum_{i=1}^{N} \mathcal{F}_h(\phi_i) \; \delta\theta_i = \sum_{i=1}^{N} F_i \, \delta\theta_i \,,$$

  where the (bi)linearity property is used, and the **coefficient matrix** (“stiffness” matrix) and **right-hand-side vector** are defined as follows:

$$A_{ij} = \mathcal{A}_h(\phi_j, \phi_i) \,, \qquad F_i = \mathcal{F}_h(\phi_i) \,.$$

Galerkin method

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- The **coefficient matrix** ("stiffness" matrix) and **right-hand-side vector** are defined as follows:

$$A_{ij} = \mathcal{A}_h(\phi_j, \phi_i) \,, \qquad F_i = \mathcal{F}_h(\phi_i) \,.$$

- Now, the approximated problem may be written as:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \Big[A_{ij} \; \theta_j - F_i \Big] \delta\theta_i = 0 \quad \forall \, \delta\theta_i.$$

Galerkin method

The interpolation and system of algebraic equations

$$u_h(x) = \sum_{j=1}^{N} \theta_j \, \phi_j(x) \,, \qquad \delta u_h(x) = \sum_{i=1}^{N} \delta\theta_i \, \phi_i(x) \,.$$

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- It is (always) true if the expression in brackets equals zero which gives the **system of algebraic equations** (for $\theta_j = ?$):

$$\sum_{i=1}^{N} A_{ij} \; \theta_j = F_i \,.$$

Galerkin method

The interpolation and system of algebraic equations

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Using this **interpolation for the approximated problem** leads to the following **system of algebraic equations** (for $\theta_j = ?$):

$$\sum_{i=1}^{N} A_{ij} \, \theta_j = F_i \,, \quad \text{where} \quad A_{ij} = \mathcal{A}_h(\phi_j, \phi_i) \,, \quad F_i = \mathcal{F}_h(\phi_i) \,.$$

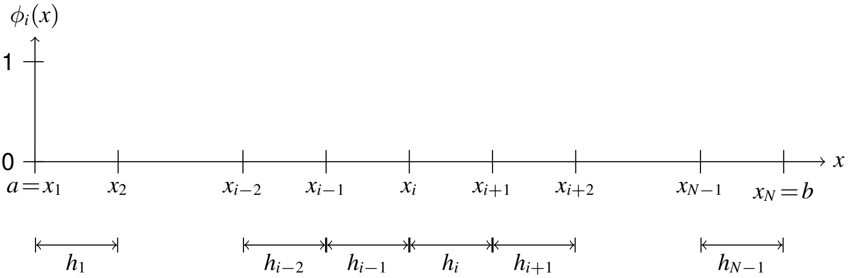
Example (for the model problem)

$$A_{ij} = \mathcal{A}_h(\phi_j, \phi_i) = \int_a^b \Big[\alpha \, \phi_i' \, \phi_j' + \gamma \, \phi_i \, \phi_j \Big] \, \mathrm{d}x \,,$$

$$F_i = \mathcal{F}_h(\phi_i) = \int_a^b f \, \phi_i \, \mathrm{d}x + \Big[\hat{q} \, \phi_i \Big]_{x=a} .$$

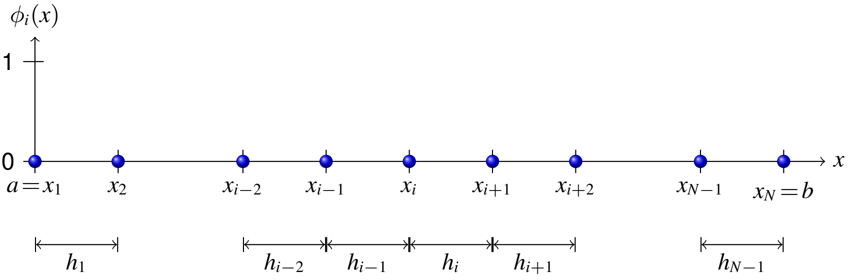
Outline

Discretization and (linear) shape functions



- The domain interval is divided into $(N-1)$ **finite elements** (subdomains).

Discretization and (linear) shape functions



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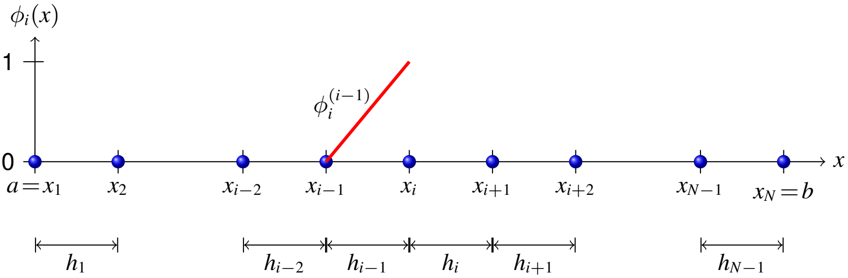

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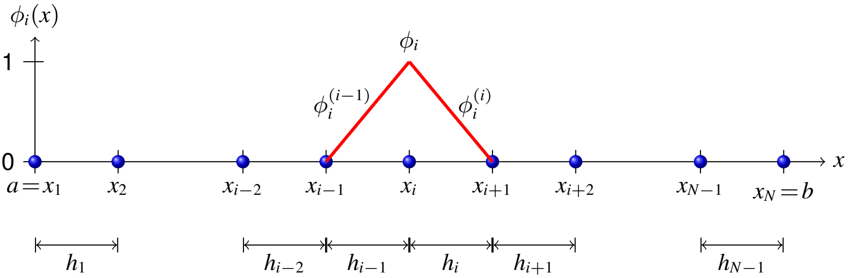
- There are N **nodes**, each with only 1 **degree of freedom (DOF)**.
- **Local (or element) shape function** is (most often) defined on an element in this way that it is equal to 1 in a particular DOF and 0 in all the others. So, there are only two *linear* interpolation functions in 1D finite element. Higher-order interpolation functions involve additional nodes (DOF) inside element.

Discretization and (linear) shape functions



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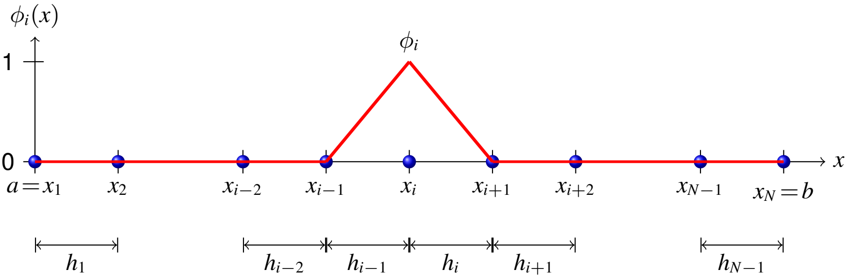

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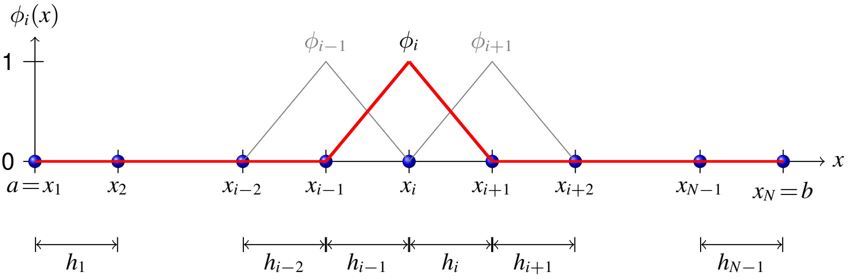
  - identically equal zero on all other elements.

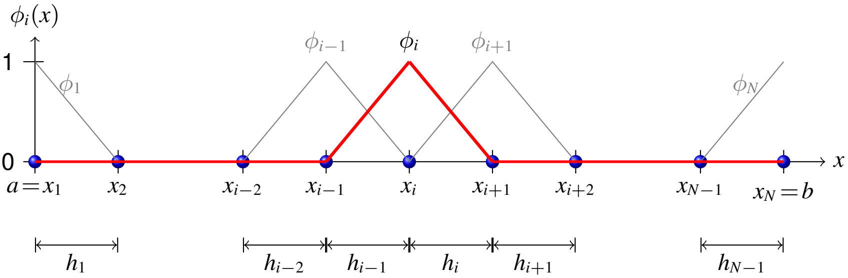
Discretization and (linear) shape functions



Shape functions for internal nodes ($i = 2, \ldots, (N-1)$) are:

$$\phi_i = \begin{cases} \dfrac{x - x_{i-1}}{h_{i-1}} & \text{for } x \in \Omega_{i-1}\,, \\ \dfrac{x_{i+1} - x}{h_i} & \text{for } x \in \Omega_i\,, \\ 0 & \text{otherwise.} \end{cases}$$

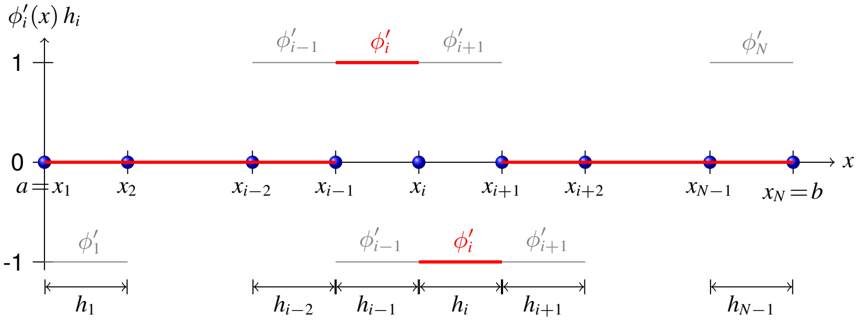
Discretization and (linear) shape functions



Shape functions for boundary nodes ($i = 1$ or N) are:

$$\phi_1 = \begin{cases} \dfrac{x_2 - x}{h_1} & \text{for } x \in \Omega_1 \,, \\ 0 & \text{otherwise,} \end{cases} \qquad \phi_N = \begin{cases} \dfrac{x - x_{N-1}}{h_{N-1}} & \text{for } x \in \Omega_{N-1} \,, \\ 0 & \text{otherwise.} \end{cases}$$

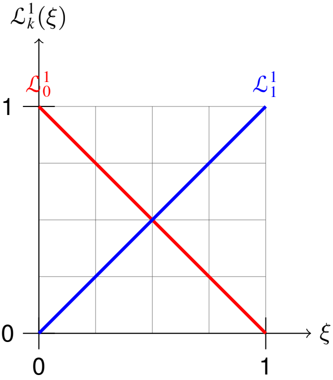
Discretization and (linear) shape functions



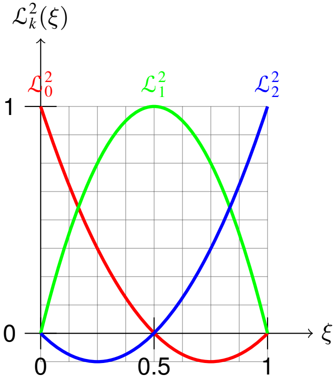
First derivatives of shape functions are discontinuous at interfaces (points) between elements (in the case of linear interpolation they are element-wise constant):

$$\phi_1' = \begin{cases} -\dfrac{1}{h_1} & \text{for } x \in \Omega_1\,, \\ 0 & \text{otherwise,} \end{cases} \qquad \phi_i' = \begin{cases} \dfrac{1}{h_{i-1}} & \text{for } x \in \Omega_{i-1}\,, \\ -\dfrac{1}{h_i} & \text{for } x \in \Omega_i\,, \\ 0 & \text{otherwise.} \end{cases} \qquad \phi_N' = \begin{cases} \dfrac{1}{h_{N-1}} & \text{for } x \in \Omega_{N-1}\,, \\ 0 & \text{otherwise.} \end{cases}$$

Lagrange interpolation functions



1st order (linear)



2nd order (quadratic)

$$\mathcal{L}_0^1(\xi) = 1 - \xi\,,$$
$$\mathcal{L}_1^1(\xi) = \xi\,,$$

$$\mathcal{L}_0^2(\xi) = (2\xi - 1)\,(\xi - 1)\,,$$
$$\mathcal{L}_1^2(\xi) = 4\xi\,(1 - \xi)\,,$$
$$\mathcal{L}_2^2(\xi) = \xi\,(2\xi - 1)\,.$$

Finite element system of algebraic equations

Matrix of the system

- The symmetry of the bilinear form $\mathcal{A}$ involves the symmetry of the matrix of the FE system of algebraic equations, i.e., $A_{ij} = A_{ji}$.

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- The product of two shape functions (or their derivatives) is nonzero only on the elements that contain the both corresponding degrees of freedom (since a shape function corresponding to a particular degree of freedom is nonzero only on the elements sharing it).
- Therefore, the integral can be computed as a sum of the integrals defined only over these finite elements that share the both degrees of freedom (since the contribution from all other elements is null):

$$A_{ij} = \sum_{e\in\mathcal{E}} A_{ij}^{(e)} = \sum_{e\in\mathcal{E}(i,j)} A_{ij}^{(e)} .$$

Here: $\mathcal{E}$ is the set of all finite elements, $\mathcal{E}(i,j)$ is the set of finite elements that contain the (both) degrees of freedom i and j.

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For a 1D problem approximated by finite elements with linear shape functions the matrix of the system will be *tridiagonal*:

$$A_{ij} = \begin{cases} A_{11}^{(1)} & \text{for } i = j = 1 \,, \\ A_{ii}^{(i-1)} + A_{ii}^{(i)} & \text{for } i = j = 2, \ldots, (N-1) \,, \\ A_{NN}^{(N-1)} & \text{for } i = j = N \,, \\ A_{i,i+1}^{(i)} & \text{for } |i-j| = 1 \,, \\ 0 & \text{for } |i-j| > 1 \,. \end{cases}$$

Finite element system of algebraic equations

Matrix of the system

For the model problem the nonzero elements of the matrix are:

$$A_{11} = \int_{x_1}^{x_2} \left[\alpha \left(\phi_1'\right)^2 + \gamma\, \phi_1^2\right] \mathrm{d}x = \int_{x_1}^{x_1+h_1} \frac{\alpha + \gamma \left(x_1 + h_1 - x\right)^2}{h_1^2} \,\mathrm{d}x\,,$$

$$A_{ii} = \int_{x_{i-1}}^{x_{i+1}} \left[\alpha \left(\phi_i'\right)^2 + \gamma\, \phi_i^2\right] \mathrm{d}x = \int_{x_i-h_{i-1}}^{x_i} \frac{\alpha + \gamma \left(x - x_i + h_{i-1}\right)^2}{h_{i-1}^2} \,\mathrm{d}x$$

$$+ \int_{x_i}^{x_i+h_i} \frac{\alpha + \gamma \left(x_i + h_i - x\right)^2}{h_i^2} \,\mathrm{d}x\,, \qquad i = 2, \ldots, (N-1)\,,$$

$$A_{NN} = \int_{x_{N-1}}^{x_N} \left[\alpha \left(\phi_N'\right)^2 + \gamma\, \phi_N^2\right] \mathrm{d}x = \int_{x_N-h_{N-1}}^{x_N} \frac{\alpha + \gamma \left(x - x_N + h_{N-1}\right)^2}{h_{N-1}^2} \,\mathrm{d}x\,,$$

$$A_{i,(i+1)} = \int_{x_i}^{x_{i+1}} \left[\alpha\, \phi_i'\, \phi_{i+1}' + \gamma\, \phi_i\, \phi_{i+1}\right] \mathrm{d}x = \int_{x_i}^{x_i+h_i} \frac{-\alpha + \gamma \left(x_i + h_i - x\right)\left(x - x_i\right)}{h_i^2} \,\mathrm{d}x\,,$$

$$i = 1, \ldots, (N-1)\,.$$

Finite element system of algebraic equations

Matrix of the system

For a homogeneous material, when $\alpha(x) = \mathrm{const} = \alpha$ and $\gamma(x) = \mathrm{const} = \gamma$, the integrals in the formulas for non-zero elements of tridiagonal matrix can be analytically integrated and the these non-zero elements are computed as follows:

$$
A_{ij} = \begin{cases}
\frac{\alpha}{h_1} + \frac{\gamma\, h_1}{3} & \text{for } i = j = 1\,, \\
\frac{\alpha}{h_{i-1}} + \frac{\gamma\, h_{i-1}}{3} + \frac{\alpha}{h_i} + \frac{\gamma\, h_i}{3} & \text{for } i = j = 2, \dots, (N-1)\,, \\
\frac{\alpha}{h_{N-1}} + \frac{\gamma\, h_{N-1}}{3} & \text{for } i = j = N\,, \\
-\frac{\alpha}{h_i} + \frac{\gamma\, h_i}{6} & \text{for } |i-j| = 1\,, \\
0 & \text{for } |i-j| > 1\,.
\end{cases}
$$

Finite element system of algebraic equations

Right-hand-side vector

The element i of the right-hand-side vector is computed as:

$$F_i = \sum_{e \in \mathcal{E}} F_i^{(e)} = \sum_{e \in \mathcal{E}(i)} F_i^{(e)} .$$

$\mathcal{E}$ is the set of all finite elements, $\mathcal{E}(i)$ is the set of finite elements that contain the degree of freedom i.

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For the considered model problem the r.h.s. vector is computed as follows:

$$F_1 = \int_{x_1}^{x_2} f\,\phi_1 \,\mathrm{d}x + \Big[\hat{q}\,\phi_1\Big]_{x=x_1} = \int_{x_1}^{x_1+h_1} \frac{f\,\big(x_1 + h_1 - x\big)}{h_1}\,\mathrm{d}x + \hat{q}\,,$$

$$F_i = \int_{x_{i-1}}^{x_{i+1}} f\,\phi_i \,\mathrm{d}x = \int_{x_i-h_{i-1}}^{x_i} \frac{f\,\big(x - x_i + h_{i-1}\big)}{h_{i-1}}\,\mathrm{d}x + \int_{x_i}^{x_i+h_i} \frac{f\,\big(x_i + h_i - x\big)}{h_i}\,\mathrm{d}x\,, \quad i = 2, \dots, (N-1)\,,$$

$F_N = ?$ (to be computed as a reaction to the essential b.c. imposed at this node)

Finite element system of algebraic equations

Right-hand-side vector

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$$F_i = \sum_{e \in \mathcal{E}} F_i^{(e)} = \sum_{e \in \mathcal{E}(i)} F_i^{(e)} .$$

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Finally, for the model problem with a uniform source (load), i.e., when $f(x) = \mathrm{const} = f$, the elements of r.h.s. vector are:

$$F_i = \begin{cases} \frac{f\,h_1}{2} + \hat{q} & \text{for } i = 1\,, \\ \frac{f\,\left(h_{i-1}+h_i\right)}{2} & \text{for } i = 2, \ldots, (N-1)\,, \\ F_N = \,? & \text{for } i = N \text{ (a reaction to the essential b.c.).} \end{cases}$$

Imposition of the essential boundary conditions

In general, the assembled matrix $[A_{ij}]$ is *singular* and the system of algebraic equations is undetermined. To make it solvable **the essential boundary conditions must be imposed**.

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Let $\mathcal{B}$ be the set of all degrees of freedom, where the essential boundary conditions are applied, that is, for $n \in \mathcal{B}$: $\theta_n = \hat{\theta}_n$, where $\hat{\theta}_n$ is a known value. In practice, the essential BCs are imposed as described below.

- Compute a new r.h.s. vector

$$\tilde{F}_i = F_i - \sum_{n \in \mathcal{B}} A_{in}\, \hat{\theta}_n \quad \text{for } i = 1, \ldots, N.$$

- Set $\tilde{F}_n = \hat{\theta}_n$.
- Set $\tilde{A}_{nn} = 1$ and all other components in the n-th row and n-th column to zero, i.e., $\tilde{A}_{ni} = \tilde{A}_{in} = \delta_{in}$ for $i = 1, \ldots, N$.
- Now, the new (sightly modified) system of equations $\boxed{\tilde{A}_{ij}\, \theta_i = \tilde{F}_j}$ is solved for θ_i.
- Finally, reactions (loads, forces) at “Dirichlet nodes” can be computed as

$$F_n = \sum_{i=1}^{N} A_{ni}\, \theta_i \,.$$

Imposition of the essential boundary conditions

For the model problem the essential b.c. are imposed only in the last node (i.e., the N-th DOF), where a known value $\hat{\theta}_N$ is given, so the modified matrix and r.h.s. vector can be formally written as follows:

$$\tilde{A}_{ij} = \begin{cases} A_{ij} & \text{for } i,j = 1,\ldots,(N-1)\,, \\ \delta_{Nj} & \text{for } i = N,\ j = 1,\ldots,N\,, \\ \delta_{iN} & \text{for } i = 1,\ldots,N,\ j = N\,, \end{cases}$$

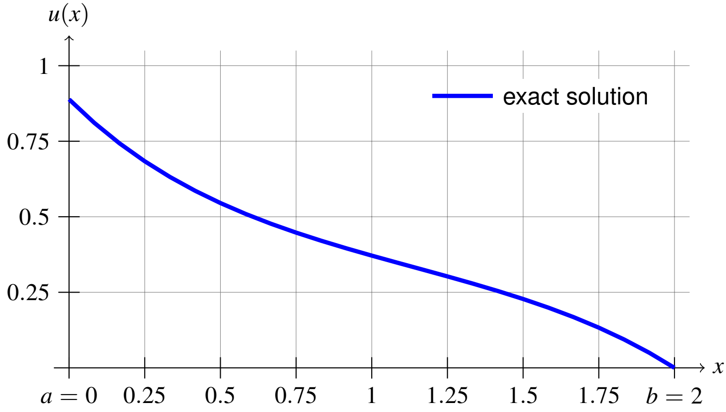
$$\tilde{F}_i = \begin{cases} F_i - A_{iN}\,\hat{\theta}_N & \text{for } i = 1,\ldots,(N-1)\,, \\ \hat{\theta}_N & \text{for } i = N\,. \end{cases}$$

After the solution of the modified system, the reaction may be computed:

$$F_N = \sum_{i=1}^{N} A_{Ni}\,\theta_i = A_{N,(N-1)}\,\theta_{N-1} + A_{NN}\,\hat{\theta}_N\,.$$

Results: analytical and FE solutions

$$\alpha(x) = 1, \qquad \gamma = 3, \qquad f(x) = 1,$$
$$a = 0, \quad q(0) = \hat{q} = 1, \qquad b = 2, \quad u(2) = \hat{u} = 0.$$



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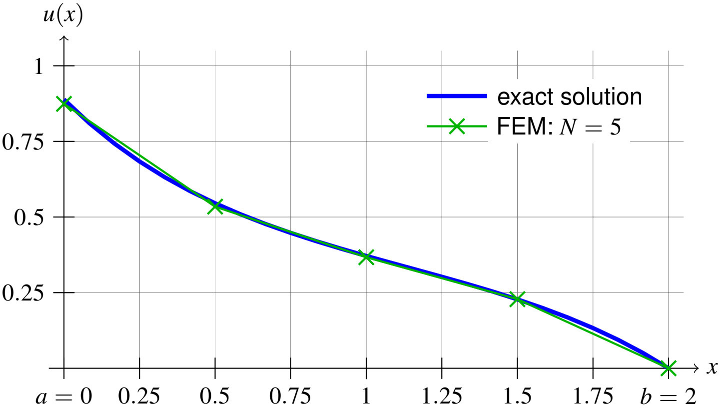

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