Mathematical Preliminaries

Introductory Course on Multiphysics Modelling

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1 Vectors, tensors, and index notation

1.1 Generalization of the concept of vector

- A **vector** is a quantity that possesses both a **magnitude** and a **direction** and obeys certain laws (of **vector algebra**):
  - the vector addition and the commutative and associative laws,
  - the associative and distributive laws for the multiplication with scalars.
- The **vectors** are suited to **describe physical phenomena**, since they are **independent of any system of reference**.

The concept of a **vector** that is independent of any coordinate system **can be generalised** to higher-order quantities, which are called **tensors**. Consequently, vectors and scalars can be treated as lower-rank tensors.

Scalars have a magnitude but no direction. They are tensors of order 0. *Example:* the mass density.

Vectors are characterised by their magnitude and direction. They are tensors of order 1. *Example:* the velocity vector.

Tensors of second order are quantities which multiplied by a vector give as the result another vector. *Example:* the stress tensor.

Higher-order tensors are often encountered in constitutive relations between second-order tensor quantities. *Example:* the fourth-order elasticity tensor.

1.2 Summation convention and index notation

Einstein's summation convention

A summation is carried out over repeated indices in an expression and the summation symbol is skipped.

Example 0.

$$
\begin{aligned}
a_i\, b_i &\equiv \sum_{i=1}^{3} a_i\, b_i = a_1\, b_1 + a_2\, b_2 + a_3\, b_3 \\
A_{ii} &\equiv \sum_{i=1}^{3} A_{ii} = A_{11} + A_{22} + A_{33} \\
A_{ij}\, b_j &\equiv \sum_{j=1}^{3} A_{ij}\, b_j = A_{i1}\, b_1 + A_{i2}\, b_2 + A_{i3}\, b_3 \quad (i = 1, 2, 3) \quad \text{[3 expressions]} \\
T_{ij}\, S_{ij} &\equiv \sum_{i=1}^{3} \sum_{j=1}^{3} T_{ij}\, S_{ij} = T_{11}\, S_{11} + T_{12}\, S_{12} + T_{13}\, S_{13} \\
&\qquad\qquad\qquad\qquad + T_{21}\, S_{21} + T_{22}\, S_{22} + T_{23}\, S_{23} \\
&\qquad\qquad\qquad\qquad + T_{31}\, S_{31} + T_{32}\, S_{32} + T_{33}\, S_{33}
\end{aligned}
$$

The principles of index notation:

- **An index cannot appear more than twice in one term!** If necessary, the standard summation symbol $(\sum)$ must be used. A repeated index is called a **bound** or **dummy index**.

Example 0.

$$A_{ii}\,, \quad C_{ijkl}\,S_{kl}\,, \quad A_{ij}\,b_i\,c_j \;\leftarrow\; \text{Correct}$$
$$A_{ij}\,b_j\,c_j \;\leftarrow\; \textbf{Wrong!}$$
$$\sum_j A_{ij}\,b_j\,c_j \;\leftarrow\; \text{Correct}$$

A term with an index repeated more than two times is correct if:

- the summation sign is used, e.g.: $\sum_i a_i\,b_i\,c_i = a_1\,b_1\,c_1 + a_2\,b_2\,c_2 + a_3\,b_3\,c_3$, or
- the dummy index is underlined, e.g.: $a_{\underline{i}}\,b_{\underline{i}}\,c_{\underline{i}} = a_1\,b_1\,c_1$ or $a_2\,b_2\,c_2$ or $a_3\,b_3\,c_3$.

- If an index appears once, it is called a **free index**. The number of free indices determines the order of a tensor.

Example 0.

$$A_{ii}\,, \quad a_i\,b_i\,, \quad T_{ij}\,S_{ij} \quad \leftarrow \quad \text{scalars (no free indices)}$$
$$A_{ij}\,b_j \quad \leftarrow \quad \text{a vector (one free index: } i\text{)}$$
$$C_{ijkl}\,S_{kl} \quad \leftarrow \quad \text{a second-order tensor (two free indices: } i, j\text{)}$$

- The denomination of dummy index (in a term) is arbitrary, since it vanishes after summation, namely: $a_i\,b_i \equiv a_j\,b_j \equiv a_k\,b_k$, etc.

Example 0.

$$a_i\,b_i = a_1\,b_1 + a_2\,b_2 + a_3\,b_3 = a_j\,b_j$$
$$A_{ii} \equiv A_{jj}\,, \quad T_{ij}\,S_{ij} \equiv T_{kl}\,S_{kl}\,, \quad T_{ij} + C_{ijkl}\,S_{kl} \equiv T_{ij} + C_{ijmn}\,S_{mn}$$

1.3 Kronecker delta and permutation symbol

Definition 0 (Kronecker delta)**.**

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

- The Kronecker delta can be used to substitute one index by another, for example: $a_i\,\delta_{ij} = a_1\,\delta_{1j} + a_2\,\delta_{2j} + a_3\,\delta_{3j} = a_j$, i.e., here $i \to j$.
- When Cartesian coordinates are used (with orthonormal base vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$) the Kronecker delta δ_{ij} is the **(matrix) representation** of the **unity tensor** $\boldsymbol{I} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 = \delta_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j$.
- $\boldsymbol{A} \bullet \boldsymbol{I} = A_{ij}\,\delta_{ij} = A_{ii}$ which is the **trace** of the matrix (tensor) $\boldsymbol{A}$.

Definition 0 (Permutation symbol)**.**

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for even permutations: 123, 231, 312} \\ -1 & \text{for odd permutations: 132, 321, 213} \\ 0 & \text{if an index is repeated} \end{cases}$$

The permutation symbol (or tensor) is widely used in index notation to express the **vector** or **cross product** of two vectors:

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \to \quad c_i = \epsilon_{ijk}\, a_j\, b_k \quad \to \quad \begin{cases} c_1 = a_2\, b_3 - a_3\, b_2 \\ c_2 = a_3\, b_1 - a_1\, b_3 \\ c_3 = a_1\, b_2 - a_2\, b_1 \end{cases}$$

1.4 Tensors and their representations

Informal definition of tensor

A **tensor** is a generalized **linear 'quantity'** that can be expressed as a **multi-dimensional array** relative to a choice of basis of the particular space on which it is defined. Therefore:

- **a tensor is independent of any chosen frame of reference,**
- **its representation behaves in a specific way under coordinate transformations.**

Cartesian system of reference

Let $\mathcal{E}^3$ be the three-dimensional **Euclidean space** with a **Cartesian coordinate system** with three **orthonormal base vectors** $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, so that

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \quad (i, j = 1, 2, 3).$$

- A **second-order tensor** $\boldsymbol{T} \in \mathcal{E}^3 \otimes \mathcal{E}^3$ is defined by

$$\begin{aligned} \boldsymbol{T} := T_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = T_{11}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + T_{12}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + T_{13}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_3 \\ + T_{21}\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + T_{22}\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + T_{23}\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_3 \\ + T_{31}\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + T_{32}\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + T_{33}\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \end{aligned}$$

  where $\otimes$ denotes the tensorial (or dyadic) product, and T_{ij} is the **(matrix) representation** of $\boldsymbol{T}$ in the given frame of reference defined by the base vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$.

- The second-order tensor $\boldsymbol{T} \in \mathcal{E}^3 \otimes \mathcal{E}^3$ can be viewed as a **linear transformation** from $\mathcal{E}^3$ onto $\mathcal{E}^3$, meaning that it transforms every vector $\boldsymbol{v} \in \mathcal{E}^3$ into another vector from $\mathcal{E}^3$ as follows

$$\begin{aligned}\boldsymbol{T}\cdot\boldsymbol{v} &= (T_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\cdot(v_k\,\boldsymbol{e}_k) = T_{ij}\,v_k\,(\overbrace{\boldsymbol{e}_j\cdot\boldsymbol{e}_k}^{\delta_{jk}})\,\boldsymbol{e}_i \\ &= T_{ij}\,v_k\,\delta_{jk}\,\boldsymbol{e}_i = T_{ij}\,v_j\,\boldsymbol{e}_i = w_i\,\boldsymbol{e}_i = \boldsymbol{w}\in\mathcal{E}^3 \quad \text{where} \quad w_i = T_{ij}\,v_j\end{aligned}$$

- A **tensor of order** n is defined by

$$\underset{n}{\boldsymbol{T}} := T_{\underbrace{ijk\ldots}_{n\text{ indices}}}\,\underbrace{\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\ldots}_{n\text{ terms}},$$

where $T_{ijk\ldots}$ is its **(n-dimensional array) representation** in the given frame of reference.

Example 0. Let $\boldsymbol{C}\in\mathcal{E}^3\otimes\mathcal{E}^3\otimes\mathcal{E}^3\otimes\mathcal{E}^3$ and $\boldsymbol{S}\in\mathcal{E}^3\otimes\mathcal{E}^3$. The fourth-order tensor $\boldsymbol{C}$ describes a linear transformation in $\mathcal{E}^3\otimes\mathcal{E}^3$:

$$\begin{aligned}\boldsymbol{C}\bullet\boldsymbol{S} = \boldsymbol{C}:\boldsymbol{S} &= (C_{ijkl}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l):(S_{mn}\,\boldsymbol{e}_m\otimes\boldsymbol{e}_n) \\ &= C_{ijkl}\,S_{mn}\,(\boldsymbol{e}_k\cdot\boldsymbol{e}_m)\,(\boldsymbol{e}_l\cdot\boldsymbol{e}_n)\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j \\ &= C_{ijkl}\,S_{mn}\,\delta_{km}\,\delta_{ln}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j = C_{ijkl}\,S_{kl}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j \\ &= T_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j = \boldsymbol{T}\in\mathcal{E}^3\otimes\mathcal{E}^3 \quad \text{where} \quad T_{ij} = C_{ijkl}\,S_{kl}\end{aligned}$$

1.5 Multiplication of vectors and tensors

Example 0. Let: s be a scalar (a zero-order tensor), $\boldsymbol{v},\boldsymbol{w}$ be vectors (first-order tensors), $\boldsymbol{R},\boldsymbol{S},\boldsymbol{T}$ be second-order tensors, $\boldsymbol{D}$ be a third-order tensor, and $\boldsymbol{C}$ be a fourth-order tensor. The order of tensors is shown explicitly in the expressions below.

$$\begin{aligned}\underset{0}{s} = \underset{1}{\boldsymbol{v}}\bullet\underset{1}{\boldsymbol{w}} = \underset{1}{\boldsymbol{v}}\,\underset{1}{\boldsymbol{w}} = \underset{1}{\boldsymbol{v}}\cdot\underset{1}{\boldsymbol{w}} &\quad\to\quad v_i\,w_i = s \\ \underset{1}{\boldsymbol{v}} = \underset{2}{\boldsymbol{T}}\,\underset{1}{\boldsymbol{w}} = \underset{2}{\boldsymbol{T}}\cdot\underset{1}{\boldsymbol{w}} &\quad\to\quad T_{ij}\,w_j = v_i \\ \underset{2}{\boldsymbol{R}} = \underset{2}{\boldsymbol{T}}\,\underset{2}{\boldsymbol{S}} = \underset{2}{\boldsymbol{T}}\cdot\underset{2}{\boldsymbol{S}} &\quad\to\quad T_{ij}\,S_{jk} = R_{ik} \\ \underset{0}{s} = \underset{2}{\boldsymbol{T}}\bullet\underset{2}{\boldsymbol{S}} = \underset{2}{\boldsymbol{T}}:\underset{2}{\boldsymbol{S}} &\quad\to\quad T_{ij}\,S_{ij} = s \\ \underset{2}{\boldsymbol{T}} = \underset{4}{\boldsymbol{C}}\bullet\underset{2}{\boldsymbol{S}} = \underset{4}{\boldsymbol{C}}:\underset{2}{\boldsymbol{S}} &\quad\to\quad C_{ijkl}\,S_{kl} = T_{ij} \\ \underset{2}{\boldsymbol{T}} = \underset{1}{\boldsymbol{v}}\,\underset{3}{\boldsymbol{D}} = \underset{1}{\boldsymbol{v}}\cdot\underset{3}{\boldsymbol{D}} &\quad\to\quad v_k\,D_{kij} = T_{ij}\end{aligned}$$

Remark: Notice a vital difference between the two dot-operators '$\bullet$' and '$\cdot$'. To avoid ambiguity, usually, the operators ':' and '$\cdot$' are not used, and the dot-operator has the meaning of the (full) dot-product, so that: $C_{ijkl}\,S_{kl}\to\boldsymbol{C}\bullet\boldsymbol{S}$, $T_{ij}\,S_{ij}\to\boldsymbol{T}\bullet\boldsymbol{S}$, and $T_{ij}\,S_{jk}\to\boldsymbol{T}\,\boldsymbol{S}$.

1.6 Vertical-bar convention and Nabla operator

Vertical-bar convention

The **vertical-bar (or comma) convention** is used to facilitate the denomination of partial derivatives with respect to the Cartesian position vectors $\boldsymbol{x} \sim x_i$, for example,

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \quad \rightarrow \quad \frac{\partial u_i}{\partial x_j} =: u_{i|j}$$

Definition 0 (Nabla-operator)**.**

$$\boxed{\nabla \equiv (.)_{|i}\, \boldsymbol{e}_i} = (.)_{|1}\, \boldsymbol{e}_1 + (.)_{|2}\, \boldsymbol{e}_2 + (.)_{|3}\, \boldsymbol{e}_3$$

The **gradient**, **divergence**, **curl (rotation)**, and **Laplacian** operations can be written using the **Nabla-operator**:

$$\begin{aligned}
\boldsymbol{v} = \operatorname{grad} s \equiv \nabla s \quad &\rightarrow \quad v_i = s_{|i} \\
\boldsymbol{T} = \operatorname{grad} \boldsymbol{v} \equiv \nabla \otimes \boldsymbol{v} \quad &\rightarrow \quad T_{ij} = v_{i|j} \\
s = \operatorname{div} \boldsymbol{v} \equiv \nabla \cdot \boldsymbol{v} \quad &\rightarrow \quad s = v_{i|i} \\
\boldsymbol{v} = \operatorname{div} \boldsymbol{T} \equiv \nabla \cdot \boldsymbol{T} \quad &\rightarrow \quad v_i = T_{ji|j} \\
\boldsymbol{w} = \operatorname{curl} \boldsymbol{v} \equiv \nabla \times \boldsymbol{v} \quad &\rightarrow \quad w_i = \epsilon_{ijk}\, v_{k|j} \\
\operatorname{lapl}(.) \equiv \Delta(.) \equiv \nabla^2(.) \quad &\rightarrow \quad (.)_{|ii}
\end{aligned}$$

Some vector calculus identities:

- $\boxed{\nabla \times (\nabla s) = \mathbf{0}}$ $\quad (\text{curl grad} = \mathbf{0})$

  Proof:

$$\nabla \times (\nabla s) = \epsilon_{ijk} \left(s_{|k}\right)_{|j} = \epsilon_{ijk}\, s_{|kj} = \begin{cases} \text{for } i = 1\text{: } s_{|23} - s_{|32} = 0 \\ \text{for } i = 2\text{: } s_{|31} - s_{|13} = 0 \\ \text{for } i = 3\text{: } s_{|12} - s_{|21} = 0 \end{cases}$$

QED

- $\boxed{\nabla \cdot (\nabla \times \boldsymbol{v}) = 0}$ $\quad (\text{div curl} = 0)$

  Proof:

$$\begin{aligned}
\nabla \cdot (\nabla \times \boldsymbol{v}) &= \left(\epsilon_{ijk}\, v_{k|j}\right)_{|i} = \epsilon_{ijk}\, v_{k|ji} \\
&= \left(v_{3|21} - v_{3|12}\right) + \left(v_{1|32} - v_{1|23}\right) + \left(v_{2|13} - v_{2|31}\right) = 0
\end{aligned}$$

QED

- $\boxed{\nabla \cdot (\nabla s) = \nabla^2 s}$ (div grad = lapl)

Proof:

$$\nabla \cdot (\nabla s) = \big(s_{|i}\big)_{|i} = s_{|ii} = s_{|11} + s_{|22} + s_{|33} \equiv \nabla^2 s$$

QED

- $\boxed{\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v}}$ (curl curl = grad div − lapl)

Proof:

$$\nabla \times (\nabla \times \boldsymbol{v}) \quad \rightarrow \quad \epsilon_{mni}\big(\epsilon_{ijk}\, v_{k|j}\big)_{|n} = \epsilon_{mni}\epsilon_{ijk}\, v_{k|jn}$$

$$\begin{aligned}
\text{for } m=1:\ \epsilon_{1ni}\epsilon_{ijk}\, v_{k|jn} &= \epsilon_{123}\big(\epsilon_{312}\, v_{2|12} + \epsilon_{321}\, v_{1|22}\big) + \epsilon_{132}\big(\epsilon_{213}\, v_{3|13} + \epsilon_{231}\, v_{1|33}\big) \\
&= \big(v_{2|2} + v_{3|3}\big)_{|1} - \big(v_{1|22} + v_{1|33}\big) \\
&= \big(v_{1|1} + v_{2|2} + v_{3|3}\big)_{|1} - \big(v_{1|11} + v_{1|22} + v_{1|33}\big) \\
&= \big(v_{i|i}\big)_{|1} - v_{1|ii} = \big(\nabla \cdot \boldsymbol{v}\big)_{|1} - \nabla^2 v_1 \\
\text{for } m=2:\ \epsilon_{2ni}\epsilon_{ijk}\, v_{k|jn} &= \big(v_{i|i}\big)_{|2} - v_{2|ii} = \big(\nabla \cdot \boldsymbol{v}\big)_{|2} - \nabla^2 v_2 \\
\text{for } m=3:\ \epsilon_{3ni}\epsilon_{ijk}\, v_{k|jn} &= \big(v_{i|i}\big)_{|3} - v_{3|ii} = \big(\nabla \cdot \boldsymbol{v}\big)_{|3} - \nabla^2 v_3
\end{aligned}$$

QED

2 Integral theorems

2.1 General idea

Integral theorems of vector calculus, namely:

- the **classical** (Kelvin-)**Stokes' theorem** (the curl theorem),
- **Green's theorem**,
- **Gauss theorem** (the Gauss-Ostrogradsky divergence theorem),

are special cases of the **general Stokes' theorem**, which generalizes the **fundamental theorem of calculus**.

Fundamental theorem of calculus relates scalar integral to boundary points:

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$$

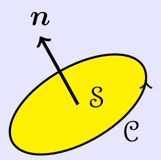
Stokes's (curl) theorem relates surface integrals to line integrals. *Applications:* for example, conservative forces.

Green's theorem is a two-dimensional special case of the Stokes' theorem.

Gauss (divergence) theorem relates volume integrals to surface integrals. *Applications:* analysis of flux, pressure.

2.2 Stokes' theorem

Theorem 1 (Stokes' curl theorem)**.** Let $\mathcal{C}$ be a simple closed curve spanned by a surface $\mathcal{S}$ with unit normal $\boldsymbol{n}$. Then, for a continuously differentiable vector field $\boldsymbol{f}$:



$$\int_{\mathcal{S}} (\nabla \times \boldsymbol{f}) \cdot \underbrace{\boldsymbol{n}\, \mathrm{d}S}_{\mathrm{d}\boldsymbol{S}} = \int_{\mathcal{C}} \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{r}$$

- *Formal requirements:* the surface $\mathcal{S}$ must be open, orientable and piecewise smooth with a correspondingly orientated, simple, piecewise and smooth boundary curve $\mathcal{C}$.
- Stokes' theorem implies that **the flux** of $\nabla \times \boldsymbol{f}$ **through a surface** $\mathcal{S}$ depends only on the boundary $\mathcal{C}$ of $\mathcal{S}$ and is therefore **independent of the surface's shape** (see Figure 1).

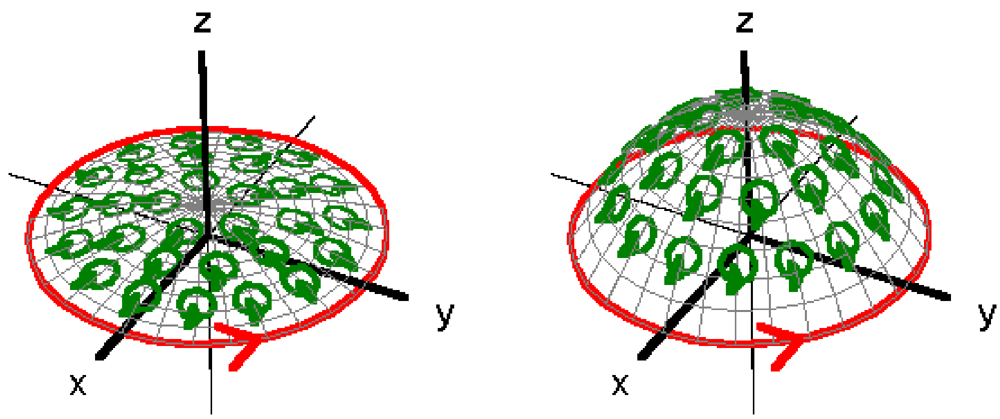


FIGURE 1

- **Green's theorem in the plane** may be viewed as a special case of Stokes' theorem (with $\boldsymbol{f} = \big[u(x, y),\ v(x, y),\ 0\big]$):

$$\int_{\mathcal{S}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y = \int_{\mathcal{C}} u\ \mathrm{d}x + v\ \mathrm{d}y$$

2.3 Gauss-Ostrogradsky theorem

Theorem 2 (Gauss divergence theorem)**.** Let the region $\mathcal{V}$ be bounded by a simple surface $\mathcal{S}$ with unit outward normal $\boldsymbol{n}$ (see Figure 2). Then, for a continuously differentiable vector field $\boldsymbol{f}$:

$$\int_{\mathcal{V}} \nabla \cdot \boldsymbol{f} \,\mathrm{d}V = \int_{\mathcal{S}} \boldsymbol{f} \cdot \underbrace{\boldsymbol{n}\,\mathrm{d}S}_{\mathrm{d}\boldsymbol{S}}; \quad \text{in particular} \quad \int_{\mathcal{V}} \nabla f \,\mathrm{d}V = \int_{\mathcal{S}} f\,\boldsymbol{n}\,\mathrm{d}S\,.$$

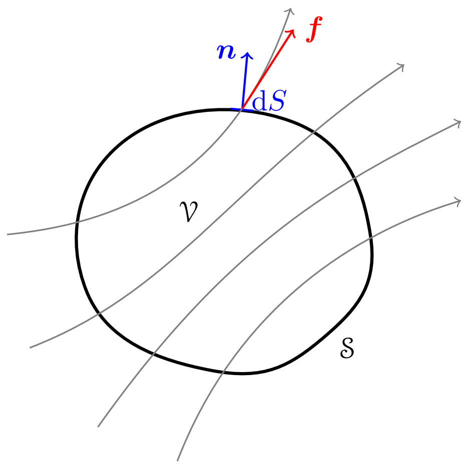


FIGURE 2

- The divergence theorem is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the vector field inside the surface.
- Intuitively, it states that the sum of all sources minus the sum of all sinks gives the net flow out of a region.

3 Time-harmonic approach

3.1 Types of dynamic problems

Dynamic problems. In dynamic problems, the field variables depend upon position $\boldsymbol{x}$ and time t, for example, $u = u(\boldsymbol{x}, t)$.

Separation of variables. In many cases, the governing PDEs can be solved by expressing u as a product of functions that each depend only on one of the independent variables: $u(\boldsymbol{x}, t) = \hat{u}(\boldsymbol{x})\,\check{u}(t)$.

Steady state. A system is in steady state if its recently observed behaviour will continue into the future. **An opposite situation is called the transient state** which is often a start-up in many steady state systems. An important case of steady state is the time-harmonic behaviour.

Time-harmonic solution. If the time-dependent function $\breve{u}(t)$ is a time-harmonic function (with the frequency f), the solution can be written as

$$u(\boldsymbol{x},t) = \hat{u}(\boldsymbol{x})\,\cos\big(\omega\,t + \alpha(\boldsymbol{x})\big)$$

where: $\omega = 2\pi\,f$ is called the **angular** (or **circular**) **frequency**, $\alpha(\boldsymbol{x})$ is the **phase-angle shift**, and $\hat{u}(\boldsymbol{x})$ can be interpreted as a **spatial amplitude**.Here:
ω – the angular frequency, $\alpha(\boldsymbol{x})$ – the phase-angle shift,
$\hat{u}(\boldsymbol{x})$ – the spatial amplitude.

3.2 Complex-valued notation

A complex-valued notation for time-harmonic problems

A convenient way to handle time-harmonic problems is in the **complex notation** with the real part (or, alternatively, the imaginary part) as a physically meaningful solution:

$$\begin{aligned} u(\boldsymbol{x},t) = \hat{u}(\boldsymbol{x})\,\cos\big(\omega\,t + \alpha(\boldsymbol{x})\big) &= \hat{u}\,\mathrm{Re}\Big\{\overbrace{\cos(\omega\,t+\alpha) + \mathrm{i}\,\sin(\omega\,t+\alpha)}^{\exp[(\mathrm{i}(\omega\,t+\alpha)]}\Big\} \\ &= \hat{u}\,\mathrm{Re}\big\{\exp[(\mathrm{i}(\omega\,t+\alpha)]\big\} = \mathrm{Re}\big\{\underbrace{\hat{u}\,\exp(\mathrm{i}\,\alpha)}_{\tilde{u}}\,\exp(\mathrm{i}\,\omega\,t)\big\} \\ &= \mathrm{Re}\big\{\tilde{u}\,\exp(\mathrm{i}\,\omega\,t)\big\} \end{aligned}$$

where the so-called **complex amplitude** (or **phasor**) is introduced:

$$\tilde{u} = \tilde{u}(\boldsymbol{x}) = \hat{u}(\boldsymbol{x})\,\exp\big(\mathrm{i}\,\alpha(\boldsymbol{x})\big) = \hat{u}(\boldsymbol{x})\big(\cos\alpha(\boldsymbol{x}) + \mathrm{i}\sin\alpha(\boldsymbol{x})\big)$$

3.3 A practical example

Consider a **linear dynamic system** characterized by the matrices of stiffness K, damping C, and mass M:

$$K\,q(t) + C\,\dot{q}(t) + M\,\ddot{q}(t) = Q(t)$$

where $Q(t)$ is the dynamic excitation (a time-varying force) and $q(t)$ is the system's response (displacement).

- Let the driving force $Q(t)$ be harmonic with the angular frequency ω and the (real-valued) amplitude $\hat{Q}$:

$$Q(t) = \hat{Q}\,\cos(\omega\,t) = \hat{Q}\,\mathrm{Re}\big\{\cos(\omega\,t) + \mathrm{i}\sin(\omega\,t)\big\} = \mathrm{Re}\big\{\hat{Q}\,\exp(\mathrm{i}\,\omega\,t)\big\}$$

- Since the system is linear the response $q(t)$ will be also harmonic and with the same angular frequency but (in general) shifted by the phase angle α:

$$\begin{aligned} q(t) = \hat{q}\,\cos(\omega\,t+\alpha) &= \hat{q}\,\mathrm{Re}\big\{\cos(\omega\,t+\alpha) + \mathrm{i}\sin(\omega\,t+\alpha)\big\} \\ &= \hat{q}\,\mathrm{Re}\big\{\exp[\mathrm{i}(\omega\,t+\alpha)]\big\} = \mathrm{Re}\big\{\underbrace{\hat{q}\,\exp(\mathrm{i}\,\alpha)}_{\tilde{q}}\,\exp(\mathrm{i}\,\omega\,t)\big\} \\ &= \mathrm{Re}\big\{\tilde{q}\,\exp(\mathrm{i}\,\omega\,t)\big\} \end{aligned}$$

Here, $\hat{q}$ and $\tilde{q}$ are the real and complex amplitudes, respectively. The real amplitude $\hat{q}$ and the phase angle α are unknowns; thus, unknown is the complex amplitude $\tilde{q} = \hat{q}\big(\cos\alpha + \mathrm{i}\sin\alpha\big)$.

- Now, one can substitute into the system's equation

$$\begin{aligned} Q(t) &\leftarrow \hat{Q}\,\exp(\mathrm{i}\,\omega\,t)\,, \\ q(t) &\leftarrow \tilde{q}\,\exp(\mathrm{i}\,\omega\,t)\,, \quad \text{so that} \quad \dot{q}(t) = \tilde{q}\,\mathrm{i}\,\omega\,\exp(\mathrm{i}\,\omega\,t)\,, \quad \ddot{q}(t) = -\tilde{q}\,\omega^2\,\exp(\mathrm{i}\,\omega\,t) \end{aligned}$$

  to obtain the following algebraic equation for the unknown complex amplitude $\tilde{q}$:

$$\big[K + \mathrm{i}\,\omega\,C - \omega^2\,M\big]\tilde{q} = \hat{Q}$$

- For the Rayleigh damping model, where $C = \beta_K\,K + \beta_M\,M$ (β_K and β_M are real-valued constants), this equation can be presented as follows:

$$\big[\tilde{K} - \omega^2\,\tilde{M}\big]\tilde{q} = \hat{Q}\,, \quad \text{where} \quad \tilde{K} = K\big(1 + \mathrm{i}\,\omega\,\beta_K\big)\,, \quad \tilde{M} = M\left(1 + \frac{\beta_M}{\mathrm{i}\,\omega}\right)$$

  are complex matrices.

- Having computed the complex amplitude $\tilde{q}$ for the given frequency ω, one can finally find the time-harmonic response as the real part of the complex solution:

$$q(t) = \mathrm{Re}\big\{\tilde{q}\,\exp(\mathrm{i}\,\omega\,t)\big\} = \hat{q}\,\cos(\omega\,t + \alpha)\,, \quad \text{where} \quad \begin{cases} \hat{q} = |\tilde{q}| \\ \alpha = \arg(\tilde{q}) \end{cases}$$

  Here, $|\tilde{q}| = \sqrt{\mathrm{Re}\{\tilde{q}\}^2 + \mathrm{Im}\{\tilde{q}\}^2}$ is the absolute value or modulus of the complex number $\tilde{q}$, and $\arg(\tilde{q}) = \arctan\left(\frac{\mathrm{Im}\{\tilde{q}\}}{\mathrm{Re}\{\tilde{q}\}}\right)$ is called the argument or angle of $\tilde{q}$.