Mathematical Preliminaries

Introductory Course on Multiphysics Modelling

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Outline

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Vectors, tensors, and index notation

Generalization of the concept of vector

- A **vector** is a quantity that possesses both a **magnitude** and a **direction** and obeys certain laws (of **vector algebra**):
  - the vector addition and the commutative and associative laws,
  - the associative and distributive laws for the multiplication with scalars.
- The **vectors** are suited to **describe physical phenomena**, since they are **independent of any system of reference**.

The concept of a **vector** that is independent of any coordinate system **can be generalised** to higher-order quantities, which are called **tensors**. Consequently, vectors and scalars can be treated as lower-rank tensors.

Vectors, tensors, and index notation

Generalization of the concept of vector

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Scalars have a magnitude but no direction. They are tensors of order 0. *Example:* the mass density.

Vectors are characterised by their magnitude and direction. They are tensors of order 1. *Example:* the velocity vector.

Tensors of second order are quantities which multiplied by a vector give as the result another vector. *Example:* the stress tensor.

Higher-order tensors are often encountered in constitutive relations between second-order tensor quantities. *Example:* the fourth-order elasticity tensor.

Vectors, tensors, and index notation

Summation convention and index notation

Einstein's summation convention

A summation is carried out over repeated indices in an expression and the summation symbol is skipped.

Example

$$a_i\,b_i \equiv \sum_{i=1}^{3} a_i\,b_i = a_1\,b_1 + a_2\,b_2 + a_3\,b_3$$

$$A_{ii} \equiv \sum_{i=1}^{3} A_{ii} = A_{11} + A_{22} + A_{33}$$

$$A_{ij}\,b_j \equiv \sum_{j=1}^{3} A_{ij}\,b_j = A_{i1}\,b_1 + A_{i2}\,b_2 + A_{i3}\,b_3 \quad (i = 1, 2, 3) \quad \text{[3 expressions]}$$

$$\begin{aligned} T_{ij}\,S_{ij} \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}\,S_{ij} &= T_{11}\,S_{11} + T_{12}\,S_{12} + T_{13}\,S_{13} \\ &+ T_{21}\,S_{21} + T_{22}\,S_{22} + T_{23}\,S_{23} \\ &+ T_{31}\,S_{31} + T_{32}\,S_{32} + T_{33}\,S_{33} \end{aligned}$$

Vectors, tensors, and index notation

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A summation is carried out over repeated indices in an expression and the summation symbol is skipped.

The principles of index notation:

- **An index cannot appear more than twice in one term!**
  If necessary, the standard summation symbol $\left(\sum\right)$ must be used. A repeated index is called a **bound** or **dummy index**.

Example

$$A_{ii}\,, \quad C_{ijkl}\,S_{kl}\,, \quad A_{ij}\,b_i\,c_j \quad \leftarrow \text{ Correct}$$

$$A_{ij}\,b_j\,c_j \quad \leftarrow \text{ Wrong!}$$

$$\sum_j A_{ij}\,b_j\,c_j \quad \leftarrow \text{ Correct}$$

A term with an index repeated more than two times is correct if:

- the summation sign is used: $\sum_i a_i\,b_i\,c_i = a_1\,b_1\,c_1 + a_2\,b_2\,c_2 + a_3\,b_3\,c_3$, or
- the dummy index is underlined: $a_{\underline{i}}\,b_{\underline{i}}\,c_{\underline{i}} = a_1\,b_1\,c_1$ or $a_2\,b_2\,c_2$ or $a_3\,b_3\,c_3$.

Vectors, tensors, and index notation

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- If an index appears once, it is called a **free index**. The number of free indices determines the order of a tensor.

Example

$$A_{ii}\,, \quad a_i\, b_i\,, \quad T_{ij}\, S_{ij} \quad \leftarrow \quad \text{scalars (no free indices)}$$

$$A_{ij}\, b_j \quad \leftarrow \quad \text{a vector (one free index: } i\text{)}$$

$$C_{ijkl}\, S_{kl} \quad \leftarrow \quad \text{a second-order tensor (two free indices: } i, j\text{)}$$

Vectors, tensors, and index notation

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- The denomination of dummy index (in a term) is arbitrary, since it vanishes after summation, namely: $a_i\, b_i \equiv a_j\, b_j \equiv a_k\, b_k$, etc.

Example

$$a_i\, b_i = a_1\, b_1 + a_2\, b_2 + a_3\, b_3 = a_j\, b_j$$

$$A_{ii} \equiv A_{jj}\,, \quad T_{ij}\, S_{ij} \equiv T_{kl}\, S_{kl}\,, \quad T_{ij} + C_{ijkl}\, S_{kl} \equiv T_{ij} + C_{ijmn}\, S_{mn}$$

Vectors, tensors, and index notation

Kronecker delta and permutation symbol

Definition (Kronecker delta)

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

Vectors, tensors, and index notation

Kronecker delta and permutation symbol

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- The Kronecker delta can be used to substitute one index by another, for example: $a_i\,\delta_{ij} = a_1\,\delta_{1j} + a_2\,\delta_{2j} + a_3\,\delta_{3j} = a_j$, i.e., here $i \to j$.
- When Cartesian coordinates are used (with orthonormal base vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$) the Kronecker delta δ_{ij} is the **(matrix) representation** of the **unity tensor** $\boldsymbol{I} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 = \delta_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j$.
- $\boldsymbol{A} \bullet \boldsymbol{I} = A_{ij}\,\delta_{ij} = A_{ii}$ which is the **trace** of the matrix (tensor) $\boldsymbol{A}$.

Vectors, tensors, and index notation

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Definition (Permutation symbol)

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for even permutations: 123, 231, 312} \\ -1 & \text{for odd permutations: 132, 321, 213} \\ 0 & \text{if an index is repeated} \end{cases}$$

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The permutation symbol (or tensor) is widely used in index notation to express the **vector** or **cross product** of two vectors:

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \rightarrow \quad c_i = \epsilon_{ijk}\, a_j\, b_k \quad \rightarrow \quad \begin{cases} c_1 = a_2\, b_3 - a_3\, b_2 \\ c_2 = a_3\, b_1 - a_1\, b_3 \\ c_3 = a_1\, b_2 - a_2\, b_1 \end{cases}$$

Vectors, tensors, and index notation

Tensors and their representations

Informal definition of tensor

A **tensor** is a generalized **linear ‘quantity’** that can be expressed as a **multi-dimensional array** relative to a choice of basis of the particular space on which it is defined. Therefore:

- **a tensor is independent of any chosen frame of reference,**
- **its representation behaves in a specific way under coordinate transformations.**

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Cartesian system of reference

Let $\mathcal{E}^3$ be the three-dimensional **Euclidean space** with a **Cartesian coordinate system** with three **orthonormal base vectors** $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, so that

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \quad (i,j = 1,2,3).$$

Vectors, tensors, and index notation

Tensors and their representations

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$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \quad (i, j = 1, 2, 3).$$

- A **second-order tensor** $\boldsymbol{T} \in \mathcal{E}^3 \otimes \mathcal{E}^3$ is defined by

$$\begin{aligned} \boldsymbol{T} := T_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = {} & T_{11}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + T_{12}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + T_{13}\, \boldsymbol{e}_1 \otimes \boldsymbol{e}_3 \\ & + T_{21}\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + T_{22}\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + T_{23}\, \boldsymbol{e}_2 \otimes \boldsymbol{e}_3 \\ & + T_{31}\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_1 + T_{32}\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_2 + T_{33}\, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \end{aligned}$$

  where $\otimes$ denotes the tensorial (or dyadic) product, and T_{ij} is the **(matrix) representation** of $\boldsymbol{T}$ in the given frame of reference defined by the base vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$.

Vectors, tensors, and index notation

Tensors and their representations

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  where $\otimes$ denotes the tensorial (or dyadic) product, and T_{ij} is the **(matrix) representation** of $\boldsymbol{T}$ in the given frame of reference defined by the base vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$.
- The second-order tensor $\boldsymbol{T} \in \mathcal{E}^3 \otimes \mathcal{E}^3$ can be viewed as a **linear transformation** from $\mathcal{E}^3$ onto $\mathcal{E}^3$, meaning that it transforms every vector $\boldsymbol{v} \in \mathcal{E}^3$ into another vector from $\mathcal{E}^3$ as follows

$$\begin{aligned}\boldsymbol{T} \cdot \boldsymbol{v} &= (T_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot (v_k\,\boldsymbol{e}_k) = T_{ij}\,v_k\,(\overbrace{\boldsymbol{e}_j \cdot \boldsymbol{e}_k}^{\delta_{jk}})\,\boldsymbol{e}_i \\ &= T_{ij}\,v_k\,\delta_{jk}\,\boldsymbol{e}_i = T_{ij}\,v_j\,\boldsymbol{e}_i = w_i\,\boldsymbol{e}_i = \boldsymbol{w} \in \mathcal{E}^3 \quad \text{where} \quad w_i = T_{ij}\,v_j\end{aligned}$$

Vectors, tensors, and index notation

Tensors and their representations

- A **tensor of order** n is defined by

$$\underset{n}{\boldsymbol{T}} := T_{\underbrace{ijk\ldots}_{n \text{ indices}}} \underbrace{\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \ldots}_{n \text{ terms}},$$

  where $T_{ijk\ldots}$ is its **(n-dimensional array) representation** in the given frame of reference.

Example

Let $\boldsymbol{C} \in \mathcal{E}^3 \otimes \mathcal{E}^3 \otimes \mathcal{E}^3 \otimes \mathcal{E}^3$ and $\boldsymbol{S} \in \mathcal{E}^3 \otimes \mathcal{E}^3$. The fourth-order tensor $\boldsymbol{C}$ describes a linear transformation in $\mathcal{E}^3 \otimes \mathcal{E}^3$:

$$\begin{aligned}
\boldsymbol{C} \bullet \boldsymbol{S} = \boldsymbol{C} : \boldsymbol{S} &= \left(C_{ijkl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l\right) : \left(S_{mn}\, \boldsymbol{e}_m \otimes \boldsymbol{e}_n\right) \\
&= C_{ijkl}\, S_{mn} \left(\boldsymbol{e}_k \cdot \boldsymbol{e}_m\right) \left(\boldsymbol{e}_l \cdot \boldsymbol{e}_n\right) \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\
&= C_{ijkl}\, S_{mn}\, \delta_{km}\, \delta_{ln}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = C_{ijkl}\, S_{kl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\
&= T_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \boldsymbol{T} \in \mathcal{E}^3 \otimes \mathcal{E}^3 \quad \text{where} \quad T_{ij} = C_{ijkl}\, S_{kl}
\end{aligned}$$

Vectors, tensors, and index notation

Multiplication of vectors and tensors

Example

Let: s be a scalar (a zero-order tensor), $\boldsymbol{v}, \boldsymbol{w}$ be vectors (first-order tensors), $\boldsymbol{R}, \boldsymbol{S}, \boldsymbol{T}$ be second-order tensors, $\boldsymbol{D}$ be a third-order tensor, and $\boldsymbol{C}$ be a fourth-order tensor. The order of tensors is shown explicitly in the expressions below.

$$
\begin{aligned}
\underset{0}{s} = \underset{1}{\boldsymbol{v}} \bullet \underset{1}{\boldsymbol{w}} = \underset{1}{\boldsymbol{v}}\,\underset{1}{\boldsymbol{w}} = \underset{1}{\boldsymbol{v}} \cdot \underset{1}{\boldsymbol{w}} \quad &\rightarrow \quad v_i\, w_i = s \\
\underset{1}{\boldsymbol{v}} = \underset{2}{\boldsymbol{T}}\,\underset{1}{\boldsymbol{w}} = \underset{2}{\boldsymbol{T}} \cdot \underset{1}{\boldsymbol{w}} \quad &\rightarrow \quad T_{ij}\, w_j = v_i \\
\underset{2}{\boldsymbol{R}} = \underset{2}{\boldsymbol{T}}\,\underset{2}{\boldsymbol{S}} = \underset{2}{\boldsymbol{T}} \cdot \underset{2}{\boldsymbol{S}} \quad &\rightarrow \quad T_{ij}\, S_{jk} = R_{ik} \\
\underset{0}{s} = \underset{2}{\boldsymbol{T}} \bullet \underset{2}{\boldsymbol{S}} = \underset{2}{\boldsymbol{T}} : \underset{2}{\boldsymbol{S}} \quad &\rightarrow \quad T_{ij}\, S_{ij} = s \\
\underset{2}{\boldsymbol{T}} = \underset{4}{\boldsymbol{C}} \bullet \underset{2}{\boldsymbol{S}} = \underset{4}{\boldsymbol{C}} : \underset{2}{\boldsymbol{S}} \quad &\rightarrow \quad C_{ijkl}\, S_{kl} = T_{ij} \\
\underset{2}{\boldsymbol{T}} = \underset{1}{\boldsymbol{v}}\,\underset{3}{\boldsymbol{D}} = \underset{1}{\boldsymbol{v}} \cdot \underset{3}{\boldsymbol{D}} \quad &\rightarrow \quad v_k\, D_{kij} = T_{ij}
\end{aligned}
$$

Remark: Notice a vital difference between the two dot-operators '$\bullet$' and '$\cdot$'. To avoid ambiguity, usually, the operators ':' and '$\cdot$' are not used, and the dot-operator has the meaning of the (full) dot-product, so that:
$C_{ijkl}\, S_{kl} \rightarrow \boldsymbol{C} \bullet \boldsymbol{S}$, $T_{ij}\, S_{ij} \rightarrow \boldsymbol{T} \bullet \boldsymbol{S}$, and $T_{ij}\, S_{jk} \rightarrow \boldsymbol{T}\,\boldsymbol{S}$.

Vectors, tensors, and index notation

Vertical-bar convention and Nabla operator

Vertical-bar convention

The **vertical-bar (or comma) convention** is used to facilitate the denomination of partial derivatives with respect to the Cartesian position vectors $\boldsymbol{x} \sim x_i$, for example,

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \quad \rightarrow \quad \frac{\partial u_i}{\partial x_j} =: u_{i|j}$$

Vectors, tensors, and index notation

Vertical-bar convention and Nabla operator

Vertical-bar convention

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \quad \rightarrow \quad \frac{\partial u_i}{\partial x_j} =: u_{i|j}$$

Definition (Nabla-operator)

$$\boxed{\nabla \equiv (.)_{|i}\, \boldsymbol{e}_i} = (.)_{|1}\, \boldsymbol{e}_1 + (.)_{|2}\, \boldsymbol{e}_2 + (.)_{|3}\, \boldsymbol{e}_3$$

The **gradient**, **divergence**, **curl (rotation)**, and **Laplacian** operations can be written using the **Nabla-operator**:

$$\begin{aligned}
\boldsymbol{v} = \operatorname{grad} s \equiv \nabla s \quad &\rightarrow \quad v_i = s_{|i} \\
s = \operatorname{div} \boldsymbol{v} \equiv \nabla \cdot \boldsymbol{v} \quad &\rightarrow \quad s = v_{i|i} \\
\boldsymbol{w} = \operatorname{curl} \boldsymbol{v} \equiv \nabla \times \boldsymbol{v} \quad &\rightarrow \quad w_i = \epsilon_{ijk}\, v_{k|j} \\
\operatorname{lapl}(.) \equiv \Delta(.) \equiv \nabla^2(.) \quad &\rightarrow \quad (.)_{|ii}
\end{aligned}$$

Vectors, tensors, and index notation

Nabla-operator and vector calculus identities

$$\boxed{\nabla \equiv (.)_{|i}\, \boldsymbol{e}_i} = (.)_{|1}\, \boldsymbol{e}_1 + (.)_{|2}\, \boldsymbol{e}_2 + (.)_{|3}\, \boldsymbol{e}_3$$

$$\begin{aligned}
\boldsymbol{v} = \operatorname{grad} s \equiv \nabla s \quad &\rightarrow \quad v_i = s_{|i} \\
\boldsymbol{T} = \operatorname{grad} \boldsymbol{v} \equiv \nabla \otimes \boldsymbol{v} \quad &\rightarrow \quad T_{ij} = v_{i|j} \\
s = \operatorname{div} \boldsymbol{v} \equiv \nabla \cdot \boldsymbol{v} \quad &\rightarrow \quad s = v_{i|i} \\
\boldsymbol{v} = \operatorname{div} \boldsymbol{T} \equiv \nabla \cdot \boldsymbol{T} \quad &\rightarrow \quad v_i = T_{ji|j} \\
\boldsymbol{w} = \operatorname{curl} \boldsymbol{v} \equiv \nabla \times \boldsymbol{v} \quad &\rightarrow \quad w_i = \epsilon_{ijk}\, v_{k|j} \\
\operatorname{lapl}(.) \equiv \Delta(.) \equiv \nabla^2(.) \quad &\rightarrow \quad (.)_{|ii}
\end{aligned}$$

Some vector calculus identities:

- $\boxed{\nabla \times (\nabla s) = \boldsymbol{0}}$ (curl grad $= \boldsymbol{0}$)
- $\boxed{\nabla \cdot (\nabla \times \boldsymbol{v}) = 0}$ (div curl $= 0$)
- $\boxed{\nabla \cdot (\nabla s) = \nabla^2 s}$ (div grad $=$ lapl)
- $\boxed{\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v}}$ (curl curl $=$ grad div $-$ lapl)

Vectors, tensors, and index notation

Vector calculus identities

- $\boxed{\nabla \times (\nabla s) = \mathbf{0}}$ (curl grad $= \mathbf{0}$)

Proof.

$$\nabla \times (\nabla s) = \epsilon_{ijk} \left(s_{|k} \right)_{|j} = \epsilon_{ijk}\, s_{|kj} = \begin{cases} \text{for } i = 1\text{:} & s_{|23} - s_{|32} = 0 \\ \text{for } i = 2\text{:} & s_{|31} - s_{|13} = 0 \\ \text{for } i = 3\text{:} & s_{|12} - s_{|21} = 0 \end{cases}$$

QED

- $\boxed{\nabla \cdot (\nabla \times \boldsymbol{v}) = 0}$ (div curl $= 0$)
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Vectors, tensors, and index notation

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- $\boxed{\nabla \times (\nabla s) = \mathbf{0}}$ (curl grad $= \mathbf{0}$)
- $\boxed{\nabla \cdot (\nabla \times \boldsymbol{v}) = 0}$ (div curl $= 0$)

Proof.

$$\begin{aligned}\nabla \cdot (\nabla \times \boldsymbol{v}) &= \left(\epsilon_{ijk}\, v_{k|j}\right)_{|i} = \epsilon_{ijk}\, v_{k|ji} \\ &= \left(v_{3|21} - v_{3|12}\right) + \left(v_{1|32} - v_{1|23}\right) + \left(v_{2|13} - v_{2|31}\right) = 0\end{aligned}$$

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Proof.

$$\nabla \cdot (\nabla s) = \left(s_{|i}\right)_{|i} = s_{|ii} = s_{|11} + s_{|22} + s_{|33} \equiv \nabla^2 s$$

QED

- $\boxed{\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v}}$ (curl curl $=$ grad div $-$ lapl)

Vectors, tensors, and index notation

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Proof.

$$\nabla \times (\nabla \times \boldsymbol{v}) \quad \rightarrow \quad \epsilon_{mni} \big(\epsilon_{ijk}\, v_{k|j}\big)_{|n} = \epsilon_{mni}\epsilon_{ijk}\, v_{k|jn}$$

for $m = 1$:
$$\begin{aligned}
\epsilon_{1ni}\epsilon_{ijk}\, v_{k|jn} &= \epsilon_{123}\big(\epsilon_{312}\, v_{2|12} + \epsilon_{321}\, v_{1|22}\big) + \epsilon_{132}\big(\epsilon_{213}\, v_{3|13} + \epsilon_{231}\, v_{1|33}\big) \\
&= \big(v_{2|2} + v_{3|3}\big)_{|1} - \big(v_{1|22} + v_{1|33}\big) \\
&= \big(v_{1|1} + v_{2|2} + v_{3|3}\big)_{|1} - \big(v_{1|11} + v_{1|22} + v_{1|33}\big) \\
&= \big(v_{i|i}\big)_{|1} - v_{1|ii} = (\nabla \cdot \boldsymbol{v})_{|1} - \nabla^2 v_1
\end{aligned}$$

for $m = 2$: $\epsilon_{2ni}\epsilon_{ijk}\, v_{k|jn} = \big(v_{i|i}\big)_{|2} - v_{2|ii} = (\nabla \cdot \boldsymbol{v})_{|2} - \nabla^2 v_2$

for $m = 3$: $\epsilon_{3ni}\epsilon_{ijk}\, v_{k|jn} = \big(v_{i|i}\big)_{|3} - v_{3|ii} = (\nabla \cdot \boldsymbol{v})_{|3} - \nabla^2 v_3$

QED

Outline

Integral theorems

General idea

Integral theorems of vector calculus, namely:

- the **classical** (Kelvin-)**Stokes' theorem** (the curl theorem),
- **Green's theorem**,
- **Gauss theorem** (the Gauss-Ostrogradsky divergence theorem),

are special cases of the **general Stokes' theorem**, which generalizes the **fundamental theorem of calculus**.

Integral theorems

General idea

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Fundamental theorem of calculus relates scalar integral to boundary points:

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$$

Stokes's (curl) theorem relates surface integrals to line integrals.
Applications: for example, conservative forces.

Green's theorem is a two-dimensional special case of the Stokes' theorem.

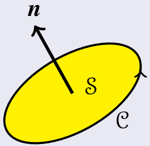
Gauss (divergence) theorem relates volume integrals to surface integrals.
Applications: analysis of flux, pressure.

Integral theorems

Stokes' theorem

Theorem (Stokes' curl theorem)

Let $\mathcal{C}$ be a simple closed curve spanned by a surface $\mathcal{S}$ with unit normal $\boldsymbol{n}$. Then, for a continuously differentiable vector field $\boldsymbol{f}$:



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Integral theorems

Stokes' theorem

Theorem (Stokes' curl theorem)

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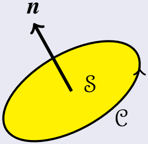

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- **Green's theorem in the plane** may be viewed as a special case of Stokes' theorem (with $\boldsymbol{f} = \left[u(x,y),\ v(x,y),\ 0\right]$):

$$\int_{\mathcal{S}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathrm{d}x\, \mathrm{d}y = \int_{\mathcal{C}} u\ \mathrm{d}x + v\ \mathrm{d}y$$

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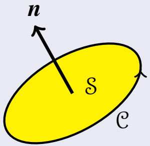

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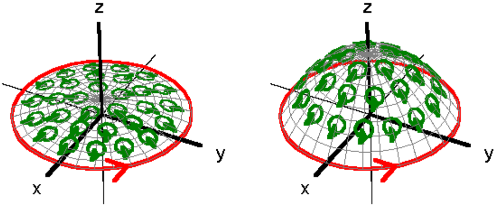
- Stokes' theorem implies that **the flux** of $\nabla \times \boldsymbol{f}$ **through a surface** $\mathcal{S}$ depends only on the boundary $\mathcal{C}$ of $\mathcal{S}$ and is therefore **independent of the surface's shape**.



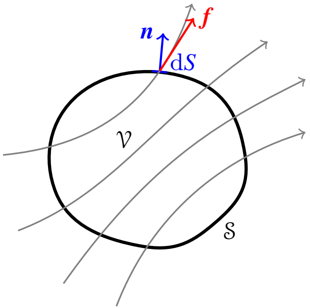
Integral theorems

Gauss-Ostrogradsky theorem

Theorem (Gauss divergence theorem)

Let the region $\mathcal{V}$ be bounded by a simple surface $\mathcal{S}$ with unit outward normal $\boldsymbol{n}$. Then, for a continuously differentiable vector field $\boldsymbol{f}$:

$$\int_{\mathcal{V}} \nabla \cdot \boldsymbol{f} \,\mathrm{d}V = \int_{\mathcal{S}} \boldsymbol{f} \cdot \underbrace{\boldsymbol{n}\,\mathrm{d}S}_{\mathrm{d}\boldsymbol{S}}; \quad \textit{in particular} \quad \int_{\mathcal{V}} \nabla f \,\mathrm{d}V = \int_{\mathcal{S}} f\,\boldsymbol{n}\,\mathrm{d}S\,.$$



- The divergence theorem is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the vector field inside the surface.
- Intuitively, it states that the sum of all sources minus the sum of all sinks gives the net flow out of a region.

Outline

Time-harmonic approach

Types of dynamic problems

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Time-harmonic approach

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Time-harmonic solution. If the time-dependent function $\check{u}(t)$ is a time-harmonic function (with the frequency f), the solution can be written as

$$u(\boldsymbol{x}, t) = \hat{u}(\boldsymbol{x}) \cos\big(\omega\, t + \alpha(\boldsymbol{x})\big)$$

where: $\omega = 2\pi f$ is called the **angular** (or **circular**) **frequency**, $\alpha(\boldsymbol{x})$ is the **phase-angle shift**, and $\hat{u}(\boldsymbol{x})$ can be interpreted as a **spatial amplitude**.

Time-harmonic approach

Complex-valued notation

Time-harmonic solution:

$$u(\boldsymbol{x},t) = \hat{u}(\boldsymbol{x})\ \cos\big(\omega\,t + \alpha(\boldsymbol{x})\big)$$

Here: ω – the angular frequency, $\alpha(\boldsymbol{x})$ – the phase-angle shift, $\hat{u}(\boldsymbol{x})$ – the spatial amplitude.

A complex-valued notation for time-harmonic problems

A convenient way to handle time-harmonic problems is in the **complex notation** with the real part as a physically meaningful solution:

$$\begin{aligned} u(\boldsymbol{x},t) &= \hat{u}(\boldsymbol{x})\ \cos\big(\omega\,t + \alpha(\boldsymbol{x})\big) = \hat{u}\ \mathrm{Re}\Big\{\overbrace{\cos(\omega\,t+\alpha) + \mathrm{i}\ \sin(\omega\,t+\alpha)}^{\exp[(\mathrm{i}(\omega\,t+\alpha)]}\Big\} \\ &= \hat{u}\ \mathrm{Re}\Big\{\exp[(\mathrm{i}(\omega\,t+\alpha)]\Big\} = \mathrm{Re}\Big\{\underbrace{\hat{u}\ \exp(\mathrm{i}\,\alpha)}_{\tilde{u}}\ \exp(\mathrm{i}\,\omega\,t)\Big\} \\ &= \mathrm{Re}\big\{\tilde{u}\ \exp(\mathrm{i}\,\omega\,t)\big\} \end{aligned}$$

where the so-called **complex amplitude** (or **phasor**) is introduced:

$$\tilde{u} = \tilde{u}(\boldsymbol{x}) = \hat{u}(\boldsymbol{x})\ \exp\big(\mathrm{i}\,\alpha(\boldsymbol{x})\big) = \hat{u}(\boldsymbol{x})\big(\cos\alpha(\boldsymbol{x}) + \mathrm{i}\sin\alpha(\boldsymbol{x})\big)$$

Time-harmonic approach

A practical example

Consider a **linear dynamic system** characterized by the matrices of stiffness K, damping C, and mass M:

$$K\,q(t) + C\,\dot{q}(t) + M\,\ddot{q}(t) = Q(t)$$

where $Q(t)$ is the dynamic excitation (a time-varying force) and $q(t)$ is the system's response (displacement).

Time-harmonic approach

A practical example

$$K\,q(t) + C\,\dot{q}(t) + M\,\ddot{q}(t) = Q(t)$$

- Let the driving force $Q(t)$ be harmonic with the angular frequency ω and the (real-valued) amplitude $\hat{Q}$:

$$Q(t) = \hat{Q}\,\cos(\omega\,t) = \hat{Q}\,\mathrm{Re}\left\{\cos(\omega\,t) + \mathrm{i}\sin(\omega\,t)\right\} = \mathrm{Re}\left\{\hat{Q}\,\exp(\mathrm{i}\,\omega\,t)\right\}$$

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- Since the system is linear the response $q(t)$ will be also harmonic and with the same angular frequency but shifted by the phase angle α:

$$\begin{aligned} q(t) &= \hat{q}\,\cos(\omega\,t+\alpha) = \hat{q}\,\mathrm{Re}\left\{\cos(\omega\,t+\alpha) + \mathrm{i}\sin(\omega\,t+\alpha)\right\} \\ &= \hat{q}\,\mathrm{Re}\left\{\exp[\mathrm{i}(\omega\,t+\alpha)]\right\} = \mathrm{Re}\left\{\underbrace{\hat{q}\,\exp(\mathrm{i}\,\alpha)}_{\tilde{q}}\,\exp(\mathrm{i}\,\omega\,t)\right\} \\ &= \mathrm{Re}\left\{\tilde{q}\,\exp(\mathrm{i}\,\omega\,t)\right\} \end{aligned}$$

  Here, $\hat{q}$ and $\tilde{q}$ are the real and complex amplitudes, respectively. The real amplitude $\hat{q}$ and the phase angle α are unknowns; thus, unknown is the complex amplitude $\tilde{q} = \hat{q}\big(\cos\alpha + \mathrm{i}\sin\alpha\big)$.

Time-harmonic approach

A practical example

$$K\, q(t) + C\, \dot{q}(t) + M\, \ddot{q}(t) = Q(t)$$

- Now, one can substitute into the system's equation

$$Q(t) \leftarrow \hat{Q}\, \exp(\mathrm{i}\,\omega\, t)\,,$$
$$q(t) \leftarrow \tilde{q}\, \exp(\mathrm{i}\,\omega\, t)\,, \quad \dot{q}(t) = \tilde{q}\,\mathrm{i}\,\omega\, \exp(\mathrm{i}\,\omega\, t)\,, \quad \ddot{q}(t) = -\tilde{q}\,\omega^2\, \exp(\mathrm{i}\,\omega\, t)$$

to obtain the following algebraic equation for the unknown complex amplitude $\tilde{q}$:

$$\left[K + \mathrm{i}\,\omega\, C - \omega^2\, M\right]\tilde{q} = \hat{Q}$$

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- For the Rayleigh damping model, where $C = \beta_K\,K + \beta_M\,M$ (β_K and β_M are real-valued constants), this equation can be presented as follows:

  $$\big[\tilde{K} - \omega^2\,\tilde{M}\big]\tilde{q} = \hat{Q}\,, \quad \text{where} \quad \tilde{K} = K\big(1 + \mathrm{i}\,\omega\,\beta_K\big)\,, \quad \tilde{M} = M\left(1 + \frac{\beta_M}{\mathrm{i}\,\omega}\right)$$

Time-harmonic approach

A practical example

$$K\,q(t) + C\,\dot{q}(t) + M\,\ddot{q}(t) = Q(t)$$

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- Having computed the complex amplitude $\tilde{q}$ for the given frequency ω, one can finally find the time-harmonic response as the real part of the complex solution:

$$q(t) = \mathrm{Re}\left\{\tilde{q}\,\exp(\mathrm{i}\,\omega\,t)\right\} = \hat{q}\,\cos(\omega\,t + \alpha)\,,\quad \text{where}\quad \begin{cases} \hat{q} = |\tilde{q}| \\ \alpha = \arg(\tilde{q}) \end{cases}$$