Ritz Method

Introductory Course on Multiphysics Modelling

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1 Introduction

1.1 Direct variational methods

Direct methods – the methods which (bypassing the derivation of the Euler equations) **go directly from a variational statement** of the problem to the solution.

Common features of these methods are presented below.

- The assumed solutions in the variational methods are in the form of **a finite linear combination of undetermined parameters with appropriately chosen functions**.
- In these methods **a continuous function is represented by a finite linear combination of functions**. However, in general, the solution of a continuum problem cannot be represented by a finite set of functions **an error is introduced** into the solution.

- The solution obtained is an approximation of the true solution for the equations describing a physical problem.
- **As the number of linearly independent terms** in the assumed solution **is increased**, **the error** in the approximation **will be reduced** (the solution converges to the desired solution)

Classical variational methods of approximation are: Ritz, Galerikin, Petrov-Galerkin (weighted residuals).

1.2 Mathematical preliminaries

Theorem 1 (Uniqueness)**.** If $\mathcal{A}$ is a strictly positive operator (i.e., $\langle \mathcal{A}\, u\,,\, u\rangle_{\mathcal{H}} > 0$ holds for all $0 \neq u \in \mathcal{D}_{\mathcal{A}}$, and $\langle \mathcal{A}\, u\,,\, u\rangle_{\mathcal{H}} = 0$ if and only if $u = 0$), then

$$\mathcal{A}\, u = f \quad \text{in } \mathcal{H}$$

has **at most** one solution $\bar{u} \in \mathcal{D}_{\mathcal{A}}$ in $\mathcal{H}$.

Proof: Suppose that there exist two solutions $\bar{u}_1, \bar{u}_2 \in \mathcal{D}_{\mathcal{A}}$. Then

$$\mathcal{A}\, \bar{u}_1 = f \quad \text{and} \quad \mathcal{A}\, \bar{u}_2 = f \quad \rightarrow \quad \mathcal{A}\left(\bar{u}_1 - \bar{u}_2\right) = 0 \quad \text{in } \mathcal{H}\,,$$

and

$$\left\langle \mathcal{A}\left(\bar{u}_1 - \bar{u}_2\right),\, \bar{u}_1 - \bar{u}_2 \right\rangle_{\mathcal{H}} = 0 \quad \rightarrow \quad \bar{u}_1 - \bar{u}_2 = 0 \quad \text{or} \quad \bar{u}_1 = \bar{u}_2\,.$$

QED

Theorem 2. Let:

- $\mathcal{A} : \mathcal{D}_{\mathcal{A}} \rightarrow \mathcal{H}$ be a positive operator (in $\mathcal{D}_{\mathcal{A}}$), and $f \in \mathcal{H}$;
- $\Pi : \mathcal{D}_{\mathcal{A}} \rightarrow \mathcal{H}$ be a quadratic functional defined as

$$\Pi(u) = \frac{1}{2}\langle \mathcal{A}\, u\,,\, u\rangle_{\mathcal{H}} - \langle f\,,\, u\rangle_{\mathcal{H}}\,.$$

1. If $\bar{u} \in \mathcal{D}_{\mathcal{A}}$ is a solution to the operator equation

$$\mathcal{A}\, u = f \quad \text{in } \mathcal{H}\,,$$

then the quadratic functional $\Pi(u)$ assumes its minimal value in $\mathcal{D}_{\mathcal{A}}$ for the element $\bar{u}$, i.e.,

$$\Pi(u) \geq \Pi(\bar{u}) \quad \text{and} \quad \Pi(u) = \Pi(\bar{u}) \text{ only for } u = \bar{u}.$$

2. Conversely, if $\Pi(u)$ assumes its minimal value, among all $u \in \mathcal{D}_{\mathcal{A}}$, for the element $\bar{u}$, then $\bar{u}$ is the solution of the operator equation, that is, $\mathcal{A}\, \bar{u} = f$.

Example (a self-adjoint operator)

- Let: $u, v \in \mathcal{H} = \{$all differentiable functions on $[0, L]\}$, $\alpha = \alpha(x)$.
- The inner (scalar) product in $\mathcal{H}$ is defined as: $\langle u\,,\, v\rangle_{\mathcal{H}} \equiv \int_0^L u\, v\, \mathrm{d}x$.
- The linear mapping $\mathcal{A} : \mathcal{H} \to \mathcal{H}$, is defined as: $\mathcal{A}(u) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha \frac{\mathrm{d}u}{\mathrm{d}x}\right)$.
  This is a **self-adjoint operator**, namely:

$$\begin{aligned}
\langle \mathcal{A}\, u\,,\, v\rangle_{\mathcal{H}} &= \int_0^L (\mathcal{A}\, u)\, v\, \mathrm{d}x = \int_0^L \left[-\frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha \frac{\mathrm{d}u}{\mathrm{d}x}\right)\right] v\ \mathrm{d}x \\
&= \left[-\alpha \frac{\mathrm{d}u}{\mathrm{d}x}\, v \right]_0^L + \int_0^L \left(\alpha \frac{\mathrm{d}u}{\mathrm{d}x}\right) \frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x = \int_0^L \alpha\, \frac{\mathrm{d}u}{\mathrm{d}x}\, \frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x \\
&= \left[\alpha\, \frac{\mathrm{d}v}{\mathrm{d}x}\, u \right]_0^L - \int_0^L u\, \frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha \frac{\mathrm{d}v}{\mathrm{d}x}\right) \mathrm{d}x \\
&= \int_0^L u\, \frac{\mathrm{d}}{\mathrm{d}x}\left(-\alpha \frac{\mathrm{d}v}{\mathrm{d}x}\right) \mathrm{d}x = \int_0^L u\,(\mathcal{A}\, v)\ \mathrm{d}x = \langle u\,,\, \mathcal{A}\, v\rangle_{\mathcal{H}}
\end{aligned}$$

2 Description of the method

2.1 Basic idea

The basic idea of the Ritz method can be presented as follows.

1. The problem must be stated in a variational form, as a **minimization problem**, that is: *find* $\bar{u}$ *minimizing certain functional* $\Pi(u)$.
2. The solution is approximated by a finite linear combination of the following form

$$\bar{u}(\boldsymbol{x}) \approx \tilde{u}^{(N)}(\boldsymbol{x}) = \sum_{j=1}^{N} c_j\, \phi_j(\boldsymbol{x}) + \phi_0(\boldsymbol{x})\,, \tag{1}$$

   where:

   c_j denote the *undetermined parameters* termed the **Ritz coefficients**,

   ϕ_0, ϕ_j are the **approximation functions** ($j = 1, \ldots, N$).
3. The parameters c_j are determined by requiring that the variational statement holds for the approximate solution, that is, $\Pi(\tilde{u}^{(N)})$ is minimized with respect to c_j ($j = 1, \ldots, N$).

Remark: The approximate solution may be exact if the set of approximation functions is well chosen (i.e., it expands a space which contains the solution).

2.2 Ritz equations for the parameters

By substituting the approximate form of solution into the functional Π one obtains Π as a *function* of the parameters c_j (after carrying out the indicated integration):

$$\Pi(\tilde{u}^{(N)}) = \tilde{\Pi}(c_1, c_2, \ldots, c_N) \tag{2}$$

The Ritz parameters are determined (or adjusted) such that $\delta\Pi = 0$. In other words, Π is minimized with respect to c_j $(j = 1, \ldots, N)$:

$$0 = \delta\Pi = \frac{\partial\Pi}{\partial c_1}\,\delta c_1 + \frac{\partial\Pi}{\partial c_2}\,\delta c_2 + \ldots + \frac{\partial\Pi}{\partial c_N}\,\delta c_N = \sum_{i=1}^{N}\frac{\partial\Pi}{\partial c_i}\,\delta c_i \tag{3}$$

Since the parameters c_j are independent, it follows that

$$\frac{\partial\Pi}{\partial c_i} = 0 \quad \text{for} \quad j = 1, \ldots, N. \tag{4}$$

These are the so-called **Ritz equations** to determine the N Ritz parameters c_j.

Quadratic functional

If the functional $\Pi(u)$ is **quadratic** in u, then its variation can be expressed as

$$\delta\Pi = \mathcal{B}(u, \delta u) - \mathcal{L}(\delta u)\,, \tag{5}$$

where $\mathcal{B}(.,.)$ and $\mathcal{L}(.)$ are certain **bilinear and linear forms**, respectively.

By applying the Ritz approximation:

$$\tilde{u}^{(N)} = \sum_{j=1}^{N}\phi_j\,c_j + \phi_0\,, \qquad \delta\tilde{u}^{(N)} = \sum_{i=1}^{N}\phi_i\,\delta c_i\,, \tag{6}$$

the following results are obtained

$$0 = \delta\Pi = \mathcal{B}(\tilde{u}, \delta\tilde{u}) - \mathcal{L}(\delta\tilde{u}) = \sum_{i=1}^{N}\left[\sum_{j=1}^{N}A_{ij}\,c_j - b_i\right]\delta c_i\,. \tag{7}$$

Now, the Ritz equations (4) form a **system of linear algebraic equations**:

$$\frac{\partial\Pi}{\partial c_j} = \sum_{j=1}^{N}A_{ij}\,c_j - b_i = 0 \quad \text{or} \quad \sum_{j=1}^{N}A_{ij}\,c_j = b_i \quad (i = 1, \ldots, N)\,. \tag{8}$$

Here, A_{ij} is the governing matrix and b_i is the right-hand-side vector.

2.3 Properties of approximation functions

The **approximation functions** must be such that the substitution of the approximate solution, $\tilde{u}^{(N)}(\boldsymbol{x})$, into the variational statement results in N linearly independent equations for the parameters c_j ($j = 1, \ldots, N$) so that the system has a solution

A convergent Ritz approximation requires the following:

1. ϕ_0 must satisfy the specified essential boundary conditions. When these conditions are homogeneous, then $\phi_0(\boldsymbol{x}) = 0$.
2. ϕ_i must satisfy the following three conditions:
   - be continuous, as required by the variational statement being used;
   - satisfy the *homogeneous form* of the specified essential boundary conditions;
   - the set $\{\phi_i\}$ must be linearly independent and complete.

If these requirements are satisfied then:

- the Ritz approximation has a **unique solution** $\tilde{u}^{(N)}(\boldsymbol{x})$,
- this **solution converges** to the true solution of the problem as the value of N is increased.

3 Simple example

3.1 Problem definition

The problem will be originally stated in the form of a Boubary-Value Problem (BVP) for an unknown scalar field to satisfy a differential equation and boundary conditions.

Consider the following **(ordinary) differential equation**

$$\textbf{(O)DE:} \quad -\frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha(x)\,\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right) = f(x) \quad \text{for } x \in (0, L) \tag{9}$$

where:

- $\alpha(x)$ and $f(x)$ are the known data of the problem: the first quantity result from the *material properties* and *geometry* of the problem whereas the second one depends on *source* or *loads*,
- $u(x)$ is the solution to be determined; it is also called **dependent variable** of the problem (with x being the **independent variable**).

The domain of this 1D problem is the interval $(0, L)$, and the points $x = 0$ and $x = L$ are the boundary points where **boundary conditions** are imposed, for example:

$$\textbf{BCs:} \quad \begin{cases} u(0) = 0 & \text{(Dirichlet b.c.)}, \\ \left[-\alpha(x)\,\frac{\mathrm{d}u}{\mathrm{d}x}(x) + k\,u(x) \right]_{x=L} = P & \text{(Robin b.c.)}. \end{cases} \tag{10}$$

Here:

- P and k are known values.

This mathematical model may describe the problem of the **axial deformation of a non-uniform elastic bar** (Fig. 1) under an axial load, fixed stiffly at one end, and subjected to an elastic spring and a force at the other end.

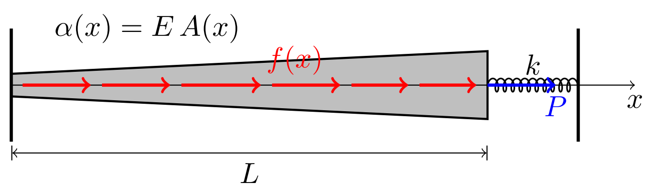

FIGURE 1: Axially loaded non-uniform elastic bar

3.2 Variational statement of the problem

▶ The Boundary-Value Problem – find $u(x)$ which satisfies ODE (9) in the domain (interval) and BCs (10) at the boundary (points) – is equivalent to minimizing the following functional:

$$\Pi(u) = \int_0^L \left[\frac{\alpha}{2} \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 - f\,u \right] \mathrm{d}x + \frac{k}{2}\big[u(L)\big]^2 - P\,u(L)\,. \tag{11}$$

This functional describes the **total potential energy** of the bar, and so the problem solution $\bar{u}$ ensures the minimum of total potential energy. The solution is to be sought among all differentiable functions satisfying the Dirichlet boundary condition (here: $u(0) = 0$).

▶ The **necessary condition** for the minimum of Π is

$$0 = \delta\Pi = \mathcal{B}(u, \delta u) - \mathcal{L}(\delta u) \quad \text{or} \quad \mathcal{B}(u, \delta u) = \mathcal{L}(\delta u)\,, \tag{12}$$

where

$$\mathcal{B}(u, \delta u) = \int_0^L \alpha\,\frac{\mathrm{d}u}{\mathrm{d}x}\,\frac{\mathrm{d}\delta u}{\mathrm{d}x}\,\mathrm{d}x + k\,u(L)\,\delta u(L)\,, \qquad \mathcal{L}(\delta u) = \int_0^L f\,\delta u\,\mathrm{d}x + P\,\delta u(L)\,. \tag{13}$$

The essential boundary condition of the problem is provided by the geometric constraint, $u(0) = 0$, and must be satisfied by $\phi_0(x)$.

3.3 Problem approximation and solution

▶ Applying the Ritz approximation and minimizing the functional results in:

$$0 = \frac{\partial \Pi}{\partial c_i} = \int_0^L \left[\alpha \, \frac{\mathrm{d}\phi_i}{\mathrm{d}x} \left(\sum_{j=1}^{N} c_j \, \frac{\mathrm{d}\phi_j}{\mathrm{d}x} + \frac{\mathrm{d}\phi_0}{\mathrm{d}x} \right) - f \, \phi_i \right] \mathrm{d}x \\ + k \, \phi_i(L) \left(\sum_{j=1}^{N} c_j \, \phi_j(L) + \phi_0(L) \right) - P \, \phi_i(L) \, , \quad (i = 1, \dots, N) \, . \tag{14}$$

That gives the system of equations for the Ritz parameters:

$$\sum_{j=1}^{N} A_{ij} \, c_j = b_i \quad (i = 1, \dots, N) \, , \tag{15}$$

$$A_{ij} = \int_0^L \alpha \, \frac{\mathrm{d}\phi_i}{\mathrm{d}x} \, \frac{\mathrm{d}\phi_j}{\mathrm{d}x} \, \mathrm{d}x + k \, \phi_i(L) \, \phi_j(L) \, , \tag{16}$$

$$b_i = -\int_0^L \left[\alpha \, \frac{\mathrm{d}\phi_i}{\mathrm{d}x} \, \frac{\mathrm{d}\phi_0}{\mathrm{d}x} - f \, \phi_i \right] \mathrm{d}x - k \, \phi_i(L) \, \phi_0(L) + P \, \phi_i(L) \, . \tag{17}$$

▶ Two approximate solutions (for $N = 1$ and $N = 2$) are presented in Table 1 for the problem data and approximation functions specified below.

- The problem data:

$$\alpha(x) = E \, A(x) = \underbrace{\alpha_0}_{E \, A_0} \left(2 - \frac{x}{L} \right) , \qquad f(x) = f_0 \, , \qquad k = 0 \, .$$

- The approximation functions:

$$\phi_0(x) = 0 \, , \qquad \phi_j(x) = x^j \text{ for } j = 1, \dots, N.$$

- The approximate solutions:

TABLE 1: The components of system matrix and right-hand side vector, and the Ritz parameters for two approximate solutions of the problem

▶ $N = 1$: $\tilde{u}^{(1)} = c_1 \, x$			**Ritz parameters**
$A_{11} = \frac{3}{2} \alpha_0 \, L$		$b_1 = \frac{1}{2} f_0 \, L^2 + P \, L$	$c_1 = \frac{f_0 \, L + 2P}{3\alpha_0}$
▶ $N = 2$: $\tilde{u}^{(2)} = c_1 \, x + c_2 \, x^2$			
$A_{11} = \frac{3}{2} \alpha_0 \, L$	$A_{12} = \frac{4}{3} \alpha_0 \, L^2$	$b_1 = \frac{1}{2} f_0 \, L^2 + P \, L$	$c_1 = \frac{7 f_0 \, L + 6P}{13\alpha_0}$
$A_{21} = A_{12}$	$A_{22} = \frac{5}{3} \alpha_0 \, L^3$	$b_2 = \frac{1}{3} f_0 \, L^3 + P \, L^2$	$c_2 = \frac{-3 f_0 \, L + 3P}{13\alpha_0 \, L}$

4 General features

General features of the Ritz method are listed below.

1. If the approximation functions satisfy the requirements, the assumed approximation $\tilde{u}^{(N)}(\boldsymbol{x})$ normally converges to the actual solution $\bar{u}(\boldsymbol{x})$ with an increase in the number of parameters, i.e., for $N \to \infty$.
2. For increasing values of N, the previously computed coefficients of the algebraic equations remain unchanged (provided the previously selected coordinate functions are not changed), and one must add newly computed coefficients to the system of equations.
3. The Ritz method applies to all problems, linear or nonlinear, as long as the variational problem is equivalent to the governing equation and natural boundary conditions.
4. If the variational problem used in the Ritz approximation is such that its bilinear form is symmetric (in u and δu), the resulting system of algebraic equations is also symmetric.
5. The governing equation and natural boundary conditions of the problem are satisfied only in the variational (integral) sense, and not in the differential equation sense.