Ritz Method

Introductory Course on Multiphysics Modelling

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Outline

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Direct variational methods

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- In these methods **a continuous function is represented by a finite linear combination of functions**. However, in general, the solution of a continuum problem cannot be represented by a finite set of functions **an error is introduced** into the solution.
- The solution obtained is an approximation of the true solution for the equations describing a physical problem.
- **As the number of linearly independent terms** in the assumed solution **is increased**, **the error** in the approximation **will be reduced** (the solution converges to the desired solution)

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Classical variational methods of approximation are: **Ritz**, **Galerikin**, **Petrov-Galerkin** (weighted residuals).

Mathematical preliminaries

Theorem (Uniqueness)

If $\mathcal{A}$ is a strictly positive operator (i.e., $\langle \mathcal{A}\, u\,,\, u\rangle_{\mathcal{H}} > 0$ holds for all $0 \neq u \in \mathcal{D}_{\mathcal{A}}$, and $\langle \mathcal{A}\, u\,,\, u\rangle_{\mathcal{H}} = 0$ if and only if $u = 0$), then

$$\mathcal{A}\, u = f \quad \text{in } \mathcal{H}$$

*has **at most** one solution $\bar{u} \in \mathcal{D}_{\mathcal{A}}$ in $\mathcal{H}$.*

SKIP PROOF

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Proof.

Suppose that there exist two solutions $\bar{u}_1, \bar{u}_2 \in \mathcal{D}_{\mathcal{A}}$. Then

$$\mathcal{A}\,\bar{u}_1 = f \quad \text{and} \quad \mathcal{A}\,\bar{u}_2 = f \quad \rightarrow \quad \mathcal{A}\left(\bar{u}_1 - \bar{u}_2\right) = 0 \quad \text{in } \mathcal{H}\,,$$

and

$$\left\langle \mathcal{A}\left(\bar{u}_1 - \bar{u}_2\right),\, \bar{u}_1 - \bar{u}_2 \right\rangle_{\mathcal{H}} = 0 \quad \rightarrow \quad \bar{u}_1 - \bar{u}_2 = 0 \quad \text{or} \quad \bar{u}_1 = \bar{u}_2\,.$$

QED

Mathematical preliminaries

Theorem

- $\mathcal{A} : \mathcal{D}_{\mathcal{A}} \to \mathcal{H}$ *be a positive operator (in* $\mathcal{D}_{\mathcal{A}}$*), and* $f \in \mathcal{H}$*;*
- $\Pi : \mathcal{D}_{\mathcal{A}} \to \mathcal{H}$ *be a quadratic functional defined as*

$$\Pi(u) = \frac{1}{2} \langle \mathcal{A}\, u \,,\, u \rangle_{\mathcal{H}} - \langle f \,,\, u \rangle_{\mathcal{H}} \;.$$

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1. *If* $\bar{u} \in \mathcal{D}_{\mathcal{A}}$ *is a solution to the operator equation*

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then the quadratic functional $\Pi(u)$ *assumes its minimal value in* $\mathcal{D}_{\mathcal{A}}$ *for the element* $\bar{u}$*, i.e.,*

$$\Pi(u) \geq \Pi(\bar{u}) \quad \text{and} \quad \Pi(u) = \Pi(\bar{u}) \text{ only for } u = \bar{u}.$$

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2. *Conversely, if* $\Pi(u)$ *assumes its minimal value, among all* $u \in \mathcal{D}_{\mathcal{A}}$*, for the element* $\bar{u}$*, then* $\bar{u}$ *is the solution of the operator equation, that is,* $\mathcal{A}\,\bar{u} = f$*.*

Mathematical preliminaries

Example: a self-adjoint operator

- Let: $u, v \in \mathcal{H} = \{\text{all differentiable functions on } [0, L]\}$, $\alpha = \alpha(x)$.
- The inner (scalar) product in $\mathcal{H}$ is defined as: $\langle u\,,\,v\rangle_{\mathcal{H}} \equiv \int_0^L u\,v\,\mathrm{d}x$.
- The linear mapping $\mathcal{A} : \mathcal{H} \to \mathcal{H}$, is defined as: $\mathcal{A}(u) \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha\,\frac{\mathrm{d}u}{\mathrm{d}x}\right)$.

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  This is a **self-adjoint operator**, namely:

$$
\begin{aligned}
\langle \mathcal{A} \, u \, , \, v \rangle_{\mathcal{H}} &= \int_0^L (\mathcal{A} \, u) \, v \, \mathrm{d}x = \int_0^L \left[- \frac{\mathrm{d}}{\mathrm{d}x} \left(\alpha \, \frac{\mathrm{d}u}{\mathrm{d}x} \right) \right] v \, \mathrm{d}x \\
&= \left[- \alpha \, \frac{\mathrm{d}u}{\mathrm{d}x} \, v \right]_0^L + \int_0^L \left(\alpha \, \frac{\mathrm{d}u}{\mathrm{d}x} \right) \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = \int_0^L \alpha \, \frac{\mathrm{d}u}{\mathrm{d}x} \, \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x \\
&= \left[\alpha \, \frac{\mathrm{d}v}{\mathrm{d}x} \, u \right]_0^L - \int_0^L u \, \frac{\mathrm{d}}{\mathrm{d}x} \left(\alpha \, \frac{\mathrm{d}v}{\mathrm{d}x} \right) \mathrm{d}x \\
&= \int_0^L u \, \frac{\mathrm{d}}{\mathrm{d}x} \left(- \alpha \, \frac{\mathrm{d}v}{\mathrm{d}x} \right) \mathrm{d}x = \int_0^L u \, (\mathcal{A} \, v) \, \mathrm{d}x = \langle u \, , \, \mathcal{A} \, v \rangle_{\mathcal{H}}
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Basic idea

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2. The solution is approximated by a finite linear combination of the following form

$$\bar{u}(\boldsymbol{x}) \approx \tilde{u}^{(N)}(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \, \phi_j(\boldsymbol{x}) + \phi_0(\boldsymbol{x}) \,,$$

where:

c_j denote the *undetermined parameters* termed the **Ritz coefficients**,

ϕ_0, ϕ_j are the **approximation functions** ($j = 1, \ldots, N$).

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   where:
   - c_j denote the *undetermined parameters* termed the **Ritz coefficients**,
   - ϕ_0, ϕ_j are the **approximation functions** ($j = 1, \ldots, N$).
3. The parameters c_j are determined by requiring that the variational statement holds for the approximate solution, that is, $\Pi(\tilde{u}^{(N)})$ is minimized with respect to c_j ($j = 1, \ldots, N$).

Remark: The approximate solution may be exact if the set of approximation functions is well chosen (i.e., it expands a space which contains the solution).

Ritz equations for the parameters

By substituting the approximate form of solution into the functional Π one obtains Π as a *function* of the parameters c_j (after carrying out the indicated integration):

$$\Pi(\tilde{u}^{(N)}) = \tilde{\Pi}(c_1, c_2, \ldots, c_N)$$

The Ritz parameters are determined (or adjusted) such that $\delta\Pi = 0$. In other words, Π is minimized with respect to c_j ($j = 1, \ldots, N$):

$$0 = \delta\Pi = \frac{\partial\Pi}{\partial c_1}\,\delta c_1 + \frac{\partial\Pi}{\partial c_2}\,\delta c_2 + \ldots + \frac{\partial\Pi}{\partial c_N}\,\delta c_N = \sum_{i=1}^{N} \frac{\partial\Pi}{\partial c_i}\,\delta c_i$$

Since the parameters c_j are independent, it follows that

$$\frac{\partial\Pi}{\partial c_i} = 0 \quad \text{for } j = 1, \ldots, N.$$

These are the so-called **Ritz equations** to determine the N Ritz parameters c_j.

Ritz equations for the parameters

Quadratic functional

If the functional $\Pi(u)$ is **quadratic** in u, then its variation can be expressed as

$$\delta\Pi = \mathcal{B}(u, \delta u) - \mathcal{L}(\delta u) \,,$$

where $\mathcal{B}(.,.)$ and $\mathcal{L}(.)$ are certain **bilinear and linear forms**, respectively.

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By applying the Ritz approximation:

$$\tilde{u}^{(N)} = \sum_{j=1}^{N} \phi_j\, c_j + \phi_0\,, \qquad \delta\tilde{u}^{(N)} = \sum_{i=1}^{N} \phi_i\, \delta c_i\,,$$

$$\Downarrow$$

$$0 = \delta\Pi = \mathcal{B}(\tilde{u}, \delta\tilde{u}) - \mathcal{L}(\delta\tilde{u}) = \sum_{i=1}^{N} \left[\sum_{j=1}^{N} A_{ij}\, c_j - b_i \right] \delta c_i\,.$$

Now, the Ritz equations form a **system of linear algebraic equations**:

$$\frac{\partial\Pi}{\partial c_j} = \sum_{j=1}^{N} A_{ij}\, c_j - b_i = 0 \quad \text{or} \quad \sum_{j=1}^{N} A_{ij}\, c_j = b_i \quad (i = 1, \ldots, N)\,.$$

Here, A_{ij} is the governing matrix and b_i is the right-hand-side vector.

Properties of approximation functions

The **approximation functions** must be such that the substitution of the approximate solution, $\tilde{u}^{(N)}(\boldsymbol{x})$, into the variational statement results in N linearly independent equations for the parameters c_j $(j = 1, \ldots, N)$ so that the system has a solution

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A convergent Ritz approximation requires the following:

1. ϕ_0 must satisfy the specified essential boundary conditions. When these conditions are homogeneous, then $\phi_0(\boldsymbol{x}) = 0$.
2. ϕ_i must satisfy the following three conditions:
   - be continuous, as required by the variational statement being used;
   - satisfy the *homogeneous form* of the specified essential boundary conditions;
   - the set $\{\phi_i\}$ must be linearly independent and complete.

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If these requirements are satisfied then:

- the Ritz approximation has a **unique solution** $\tilde{u}^{(N)}(\boldsymbol{x})$,
- this **solution converges** to the true solution of the problem as the value of N is increased.

Outline

Problem definition

(O)DE: $$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha(x)\,\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right) = f(x) \quad \text{for } x \in (0, L)$$

- $\alpha(x)$ and $f(x)$ are the known data of the problem: the first quantity result from the *material properties* and *geometry* of the problem whereas the second one depends on *source* or *loads*,
- $u(x)$ is the solution to be determined; it is also called **dependent variable** of the problem (with x being the **independent variable**).

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The domain of this 1D problem is the interval $(0,L)$, and the points $x=0$ and $x=L$ are the boundary points where **boundary conditions** are imposed, for example:

BCs: $$\begin{cases} u(0)=0 & \text{(Dirichlet b.c.)}, \\ \left[-\alpha(x)\,\frac{\mathrm{d}u}{\mathrm{d}x}(x)+k\,u(x)\right]_{x=L}=P & \text{(Robin b.c.)}. \end{cases}$$

- P and k are known values.

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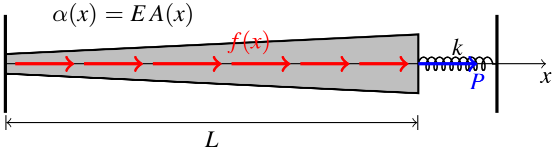
This mathematical model may describe the problem of the **axial deformation of a non-uniform elastic bar** under an axial load, fixed stiffly at one end, and subjected to an elastic spring and a force at the other end.



Variational statement of the problem

- The Boundary-Value Problem – find $u(x)$ which satisfies ODE+BCs – is equivalent to **minimizing the following functional**:

$$\Pi(u) = \int_0^L \left[\frac{\alpha}{2} \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 - f\,u \right] \mathrm{d}x + \frac{k}{2} \left[u(L) \right]^2 - P\,u(L)\,.$$

This functional describes the **total potential energy** of the bar, and so the problem solution $\bar{u}$ ensures the minimum of total potential energy.

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This functional describes the **total potential energy** of the bar, and so the problem solution $\bar{u}$ ensures the minimum of total potential energy.

- The **necessary condition** for the minimum of Π is

$$0 = \delta\Pi = \mathcal{B}(u,\delta u) - \mathcal{L}(\delta u) \quad \text{or} \quad \mathcal{B}(u,\delta u) = \mathcal{L}(\delta u)\,,$$

where

$$\mathcal{B}(u,\delta u) = \int_0^L \alpha\, \frac{\mathrm{d}u}{\mathrm{d}x}\, \frac{\mathrm{d}\delta u}{\mathrm{d}x}\, \mathrm{d}x + k\,u(L)\,\delta u(L)\,, \qquad \mathcal{L}(\delta u) = \int_0^L f\,\delta u\, \mathrm{d}x + P\,\delta u(L)\,.$$

The essential boundary condition of the problem is provided by the geometric constraint, $u(0) = 0$, and must be satisfied by $\phi_0(x)$.

Problem approximation and solution

- Applying the Ritz approximation and minimizing the functional results in:

$$0 = \frac{\partial \Pi}{\partial c_i} = \int_0^L \left[\alpha \, \frac{\mathrm{d}\phi_i}{\mathrm{d}x} \left(\sum_{j=1}^{N} c_j \, \frac{\mathrm{d}\phi_j}{\mathrm{d}x} + \frac{\mathrm{d}\phi_0}{\mathrm{d}x} \right) - f \, \phi_i \right] \mathrm{d}x$$

$$+ k \, \phi_i(L) \left(\sum_{j=1}^{N} c_j \, \phi_j(L) + \phi_0(L) \right) - P \, \phi_i(L) \,, \quad (i = 1, \ldots, N) \,.$$

That gives the **system of equations for the Ritz parameters**:

$$\sum_{j=1}^{N} A_{ij} \, c_j = b_i \quad (i = 1, \ldots, N) \,,$$

$$A_{ij} = \int_0^L \alpha \, \frac{\mathrm{d}\phi_i}{\mathrm{d}x} \, \frac{\mathrm{d}\phi_j}{\mathrm{d}x} \, \mathrm{d}x + k \, \phi_i(L) \, \phi_j(L) \,,$$

$$b_i = - \int_0^L \left[\alpha \, \frac{\mathrm{d}\phi_i}{\mathrm{d}x} \, \frac{\mathrm{d}\phi_0}{\mathrm{d}x} - f \, \phi_i \right] \mathrm{d}x - k \, \phi_i(L) \, \phi_0(L) + P \, \phi_i(L) \,.$$

Problem approximation and solution

- The problem data:

$$\alpha(x) = EA(x) = \underbrace{\alpha_0}_{EA_0}\left(2 - \frac{x}{L}\right), \qquad f(x) = f_0\,, \qquad k = 0\,.$$

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- The approximate solutions:

▶ $N = 1$: $\tilde{u}^{(1)} = c_1\,x$		**Ritz parameters**
$A_{11} = \frac{3}{2}\,\alpha_0\,L$	$b_1 = \frac{1}{2}\,f_0\,L^2 + P\,L$	$c_1 = \frac{f_0\,L + 2P}{3\alpha_0}$
▶ $N = 2$: $\tilde{u}^{(2)} = c_1\,x + c_2\,x^2$		
$A_{11} = \frac{3}{2}\,\alpha_0\,L$ $\quad A_{12} = \frac{4}{3}\,\alpha_0\,L^2$	$b_1 = \frac{1}{2}\,f_0\,L^2 + P\,L$	$c_1 = \frac{7f_0\,L + 6P}{13\alpha_0}$
$A_{21} = A_{12}$ $\quad A_{22} = \frac{5}{3}\,\alpha_0\,L^3$	$b_2 = \frac{1}{3}\,f_0\,L^3 + P\,L^2$	$c_2 = \frac{-3f_0\,L + 3P}{13\alpha_0\,L}$

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General features of the method

1. If the **approximation functions satisfy the requirements**, the assumed **approximation** $\tilde{u}^{(N)}(\boldsymbol{x})$ normally **converges to the actual solution** $\bar{u}(\boldsymbol{x})$ with an increase in the number of parameters, i.e., for $N \to \infty$.

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4. If the variational problem is such that its **bilinear form is symmetric** (in u and δu), the resulting **system of algebraic equations is also symmetric**.
5. The governing equation and natural boundary conditions of the problem are **satisfied only in the variational (integral) sense**, and not in the differential equation sense.