Fundamentals of Fluid Dynamics: Elementary Viscous Flow

Introductory Course on Multiphysics Modelling

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1.1 Newtonian fluids and viscosity

Definition 0 (Newtonian fluid). A **Newtonian fluid** is a viscous fluid for which the shear stress is proportional to the velocity gradient (i.e., to the rate of strain):

$$\tau = \mu \, \frac{\mathrm{d}u}{\mathrm{d}y} \, . \tag{1}$$

Here: τ $[\mathrm{Pa}]$ is the shear stress (“drag”) exerted by the fluid,
μ $[\mathrm{Pa} \cdot \mathrm{s}]$ is the (**dynamic** or **absolute**) **viscosity**,
$\frac{\mathrm{d}u}{\mathrm{d}y}$ $\left[\frac{1}{\mathrm{s}}\right]$ is the velocity gradient perpendicular to the direction of shear.

Figure 1 shows some exemplary velocity profiles with the corresponding distributions of shear stresses in Newtonian fluids.

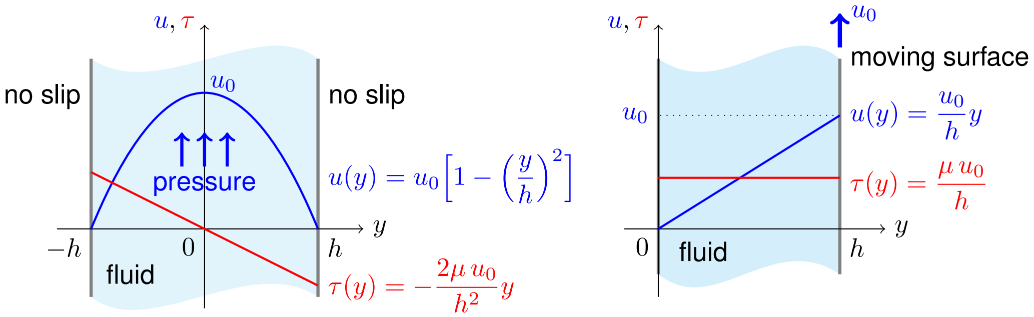

FIGURE 1: Examples of velocity and shear-stress profiles in Newtonian fluid: (*left*) the fluid under pressure flowing in a slit, (*right*) the fluid in motion by a moving surface.

Very often the ratio of the viscous force to the inertial force (the latter characterised by the fluid density ϱ) plays an important role; thus, the so-called **kinematic viscosity** ν (defined below) is more significant and informative than the absolute viscosity μ.

Definition 0 (Kinematic viscosity)**.** The **kinematic viscosity** of a fluid is defined as the quotient of its absolute viscosity μ and density ϱ:

$$\nu = \frac{\mu}{\varrho} \quad \left[\frac{\mathrm{m}^2}{\mathrm{s}}\right] . \tag{2}$$

Table 1 compares dynamic and kinematic viscosities for air and water at 20°C.

TABLE 1: Dynamic and kinematic viscosities for air and water.

fluid	μ $[10^{-5}\mathrm{Pa} \cdot \mathrm{s}]$	ν $[10^{-5}\mathrm{m}^2/\mathrm{s}]$
air (at 20°C)	1.82	1.51
water (at 20°C)	100.2	0.1004

Non-Newtonian fluids

For a non-Newtonian fluid the **viscosity changes with the applied strain rate** (velocity gradient). As a result, non-Newtonian fluids may not have a well-defined viscosity.

1.2 Constitutive relation for Newtonian fluids

The **stress tensor** can be decomposed into **spherical** and **deviatoric parts**:

$$\boldsymbol{\sigma} = \boldsymbol{\tau} - p\,\boldsymbol{I} \quad \text{or} \quad \sigma_{ij} = \tau_{ij} - p\,\delta_{ij}\,, \quad \text{where} \quad p = -\frac{1}{3}\operatorname{tr}\boldsymbol{\sigma} = -\frac{1}{3}\sigma_{ii} \tag{3}$$

is the (mechanical) **pressure** and $\boldsymbol{\tau}$ is the the **stress deviator** (**shear stress tensor**).

Using this decomposition Stokes (1845) deduced his constitutive relation for Newtonian fluids from three elementary hypotheses:

1. $\boldsymbol{\tau}$ should be **linear** function of the **velocity gradient**;
2. this relationship should be **isotropic**, as the physical properties of the fluid are assumed to show **no preferred direction**;
3. $\boldsymbol{\tau}$ should **vanish** if the flow involves **no deformation** of fluid elements.

Moreover, the **principle of conservation of moment of momentum** implies the **symmetry of stress tensor**: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}$, i.e., $\sigma_{ij} = \sigma_{ji}$. Therefore, the stress deviator $\boldsymbol{\tau}$ should also be symmetric: $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathsf{T}}$, i.e., $\tau_{ij} = \tau_{ji}$ (since the spherical part is always symmetric).

Constitutive relation for Newtonian fluids

$$\boldsymbol{\sigma} = \underbrace{\mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}\right)}_{\boldsymbol{\tau}\text{ for incompressible}} - p\,\boldsymbol{I} \quad \text{or} \quad \sigma_{ij} = \mu\left(u_{i|j} + u_{j|i}\right) - p\,\delta_{ij}\,. \tag{4}$$

This is a relation for incompressible fluid (i.e., when $\nabla\cdot\boldsymbol{u} = 0$).

1.3 Constitutive relation for compressible viscous flow

Definition 0 (Rate of strain)**.**

$$\dot{\boldsymbol{\varepsilon}} = \tfrac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}\right) \tag{5}$$

The deviatoric (shear) and volumetric strain rates are given as $\left(\dot{\boldsymbol{\varepsilon}} - \tfrac{1}{3}(\operatorname{tr}\dot{\boldsymbol{\varepsilon}})\,\boldsymbol{I}\right)$ and $\operatorname{tr}\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}\cdot\boldsymbol{I} = \nabla\cdot\boldsymbol{u}$, respectively.

Newtonian fluids are characterized by a linear, **isotropic relation** between stresses and strain rates. That requires **two constants**:

- the **viscosity** μ – to relate the deviatoric (shear) stresses to the deviatoric (shear) strain rates:

$$\boldsymbol{\tau} = 2\mu\Big(\dot{\boldsymbol{\varepsilon}} - \tfrac{1}{3}(\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}})\,\boldsymbol{I}\Big)\,, \tag{6}$$

- the so-called **volumetric viscosity** κ – to relate the mechanical pressure (the mean stress) to the volumetric strain rate:

$$p \equiv -\tfrac{1}{3}\,\mathrm{tr}\,\boldsymbol{\sigma} = -\kappa\;\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}} + p_0\,. \tag{7}$$

Here, p_0 is the **initial hydrostatic pressure** independent of the strain rate.

Volumetric viscosity

There is little evidence about the existence of volumetric viscosity and Stokes made the hypothesis that $\boxed{\kappa = 0}$. This is frequently used though it has not been definitely confirmed.

Constitutive relation for compressible Newtonian fluids

$$\boldsymbol{\sigma} = 2\mu\Big(\dot{\boldsymbol{\varepsilon}} - \tfrac{1}{3}(\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}})\,\boldsymbol{I}\Big) - p\,\boldsymbol{I} = 2\mu\,\dot{\boldsymbol{\varepsilon}} - \Big(p + \tfrac{2}{3}\mu\;\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}}\Big)\boldsymbol{I}\,, \tag{8}$$

and after using the definition for strain rate:

$$\boldsymbol{\sigma} = \mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}\right) - \left(p + \tfrac{2}{3}\mu\,\nabla\cdot\boldsymbol{u}\right)\boldsymbol{I} \quad \text{or} \quad \sigma_{ij} = \mu\left(u_{i|j} + u_{j|i}\right) - \left(p + \tfrac{2}{3}\mu\,u_{k|k}\right)\delta_{ij}\,. \tag{9}$$

2 Navier–Stokes equations

2.1 Continuity equation

Continuity (or mass conservation) equation

The balance of mass flow entering and leaving an infinitesimal control volume is equal to the rate of change in density:

$$\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\,\nabla\cdot\boldsymbol{u} = \frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\,\boldsymbol{u}) = 0\,. \tag{10}$$

For **incompressible flows** the density does not change ($\varrho = \varrho_0$ where ϱ_0 is the constant initial density) so

$$\frac{\mathrm{D}\varrho}{\mathrm{D}t} = 0 \qquad \rightarrow \qquad \nabla\cdot\boldsymbol{u} = 0\,. \tag{11}$$

This last **kinematic constraint** for the velocity field is called the **incompressibility condition**.

2.2 Cauchy's equation of motion

The general **equation of motion** valid for any continuous medium is obtained from the **principle of conservation of linear momentum**:

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}} \varrho\,\boldsymbol{u}\;\mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{b}\;\mathrm{d}\mathcal{V} + \int_{\mathcal{S}} \boldsymbol{t}\;\mathrm{d}\mathcal{S} \tag{12}$$

where $\boldsymbol{b}$ is the body (or volume) force, and $\boldsymbol{t}$ is the surface traction.

- Use the **Reynolds' transport theorem**

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}} f\;\mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \left(\frac{\mathrm{D}f}{\mathrm{D}t} + f\,\nabla\cdot\boldsymbol{u}\right)\mathrm{d}\mathcal{V}\,, \tag{13}$$

and the **continuity equation** (10)

$$\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\,\nabla\cdot\boldsymbol{u} = 0\,, \tag{14}$$

for the inertial term:

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathcal{V}} \varrho\,\boldsymbol{u}\;\mathrm{d}\mathcal{V} &= \int_{\mathcal{V}} \left[\frac{\mathrm{D}(\varrho\,\boldsymbol{u})}{\mathrm{D}t} + \varrho\,\boldsymbol{u}\,\nabla\cdot\boldsymbol{u}\right]\mathrm{d}\mathcal{V}\\ &= \int_{\mathcal{V}} \left[\varrho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{u}\underbrace{\left(\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\,\nabla\cdot\boldsymbol{u}\right)}_{0}\right]\mathrm{d}\mathcal{V} = \int_{\mathcal{V}} \varrho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\;\mathrm{d}\mathcal{V}\,.\end{aligned} \tag{15}$$

- Apply the **Cauchy's formula**: $\boldsymbol{t} = \boldsymbol{\sigma}\cdot\boldsymbol{n}$, and the **divergence theorem** for the surface traction term:

$$\int_{\mathcal{S}} \boldsymbol{t}\;\mathrm{d}\mathcal{S} = \int_{\mathcal{S}} \boldsymbol{\sigma}\cdot\boldsymbol{n}\;\mathrm{d}\mathcal{S} = \int_{\mathcal{V}} \nabla\cdot\boldsymbol{\sigma}\;\mathrm{d}\mathcal{V}\,. \tag{16}$$

- Now, the **global (integral) form** of equation of motion is obtained:

$$\int_{\mathcal{V}} \left(\varrho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} - \nabla\cdot\boldsymbol{\sigma} - \boldsymbol{b}\right)\mathrm{d}\mathcal{V} = 0\,, \tag{17}$$

which, being true for arbitrary $\mathcal{V}$ and provided that the integrand is continuous, yields the **local (differential) form** – the Cauchy's equation of motion.

Cauchy's equation of motion

$$\varrho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla\cdot\boldsymbol{\sigma} + \boldsymbol{b} \quad \text{or} \quad \varrho\,\frac{\mathrm{D}u_i}{\mathrm{D}t} = \sigma_{ij|j} + b_i \tag{18}$$

2.3 Navier–Stokes equations of motion

On applying the constitutive relations of Newtonian incompressible fluids to the Cauchy's equation of motion of continuous media, the so-called **incompressible Navier–Stokes equations** are obtained.

Incompressible Navier–Stokes equations

$$\varrho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \mu \triangle \boldsymbol{u} - \nabla p + \varrho_0 \boldsymbol{g} \quad \text{or} \quad \varrho_0 \frac{\mathrm{D}u_i}{\mathrm{D}t} = \mu \, u_{i|jj} - p_{|i} + \varrho_0 \, g_i \,, \tag{19}$$

$$\text{(+ the incompressibility constraint:)} \quad \nabla \cdot \boldsymbol{u} = 0 \quad \text{or} \quad u_{i|i} = 0 \,. \tag{20}$$

Here, the density is constant $\varrho = \varrho_0$, and the body force $\boldsymbol{b}$ has been substituted by the the gravitational force $\varrho_0 \, \boldsymbol{g}$, where $\boldsymbol{g}$ is the gravitaty acceleration. Now, on dividing by ϱ_0, using $\nu = \frac{\mu}{\varrho_0}$, and expanding the total-time derivative the main relations can be written as

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} = \nu \triangle \boldsymbol{u} - \frac{1}{\varrho_0} \nabla p + \boldsymbol{g} \quad \text{or} \quad \frac{\partial u_i}{\partial t} + u_j \, u_{i|j} = \nu \, u_{i|jj} - \frac{1}{\varrho_0} p_{|i} + g_i \,. \tag{21}$$

They differ from the Euler equations by virtue of the **viscous term**.

2.4 Boundary conditions (for incompressible flow)

- Let $\boldsymbol{n}$ be the unit normal vector to the boundary, and $\boldsymbol{m}^{(1)}$, $\boldsymbol{m}^{(2)}$ be two (non-parallel) unit tangential vectors.
- Let $\hat{\boldsymbol{u}}$, $\hat{u}_n$, $\hat{p}$ be values prescribed on the boundary, namely, the prescribed velocity vector, normal velocity, and pressure, respectively.
- Now, the following conditions can be specified on the boundaries of fluid domain.

Inflow/Outflow velocity or No-slip condition:

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \quad (\hat{\boldsymbol{u}} = \boldsymbol{0} \text{ for the no-slip condition}) \,. \tag{22}$$

Slip or Symmetry condition:

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{n} = \hat{u}_n \quad (\hat{u}_n = 0 \text{ for the symmetry condition}) \,, \\ (\boldsymbol{\sigma} \, \boldsymbol{n}) \cdot \boldsymbol{m}^{(1)} = 0 \,, \quad (\boldsymbol{\sigma} \, \boldsymbol{n}) \cdot \boldsymbol{m}^{(2)} = 0 \\ \qquad \text{(or:} \quad (\boldsymbol{\tau} \, \boldsymbol{n}) \cdot \boldsymbol{m}^{(1)} = 0 \,, \quad (\boldsymbol{\tau} \, \boldsymbol{n}) \cdot \boldsymbol{m}^{(2)} = 0) \,. \end{cases} \tag{23}$$

Pressure condition:

$$\boldsymbol{\sigma} \, \boldsymbol{n} = -\hat{p} \, \boldsymbol{n} \quad \text{(or:} \quad p = \hat{p} \,, \quad \boldsymbol{\tau} \, \boldsymbol{n} = \boldsymbol{0}) \,. \tag{24}$$

Normal flow:

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{m}^{(1)} = 0 \,, \quad \boldsymbol{u} \cdot \boldsymbol{m}^{(2)} = 0 \,, \\ (\boldsymbol{\sigma} \, \boldsymbol{n}) \cdot \boldsymbol{n} = -\hat{p} \quad \text{(or:} \quad p = \hat{p} \,, \quad (\boldsymbol{\tau} \, \boldsymbol{n}) \cdot \boldsymbol{n} = 0) \,. \end{cases} \tag{25}$$

2.5 Compressible Navier–Stokes equations of motion

On applying the constitutive relations of Newtonian compressible flow to the Cauchy's equation of motion, the **compressible Navier–Stokes equations of motion** are obtained.

Compressible Navier–Stokes equations of motion

$$\varrho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \mu \triangle \boldsymbol{u} + \frac{\mu}{3}\nabla(\nabla \cdot \boldsymbol{u}) - \nabla p + \varrho \boldsymbol{g} \quad \text{or} \quad \varrho \frac{\mathrm{D}u_i}{\mathrm{D}t} = \mu\, u_{i|jj} + \frac{\mu}{3} u_{j|ji} - p_{|i} + \varrho\, g_i \tag{26}$$

$$\text{(+ the continuity equation:)} \quad \frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho \nabla \cdot \boldsymbol{u} = 0 \quad \text{or} \quad \frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\, u_{i|i} = 0\,. \tag{27}$$

- These equations are **incomplete** – there are only 4 relations for 5 unknown fields: ϱ, $\boldsymbol{u}$, p.
- They can be completed by a **state relationship between ϱ and p**.
- However, this would normally introduce also another state variable: the temperature T, and that would involve the requirement for energy balance (yet another equations). Such approach is governed by the **complete Navier–Stokes equations for compressible flow**.
- More simplified yet complete set of equations can be used to describe an **isothermal flow with small compressibility**.

2.6 Small-compressibility Navier–Stokes equations

Assumptions:

1. The problem is **isothermal**.
2. The **variation of ϱ with p is very small**, such that *in* product terms of $\boldsymbol{u}$ and ϱ the latter can be assumed constant: $\varrho = \varrho_0$.

Small compressibility is allowed: density changes are, as a consequence of elastic deformability, related to pressure changes:

$$\mathrm{d}\varrho = \frac{\varrho_0}{K}\,\mathrm{d}p \quad \to \quad \frac{\partial \varrho}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} \quad \text{where } c = \sqrt{\frac{K}{\varrho_0}} \tag{28}$$

is the acoustic wave velocity, and K is the elastic bulk modulus. This relation can be used for the continuity equation yielding the following small-compressibility equation:

$$\frac{\partial p}{\partial t} = -\underbrace{c^2 \varrho_0}_{K}\, \nabla \cdot \boldsymbol{u}\,, \tag{29}$$

where the density term standing by $\boldsymbol{u}$ has been assumed constant: $\varrho = \varrho_0$. This also applies now to the Navier-Stokes momentum equations of compressible flow ($\nu = \frac{\mu}{\varrho_0}$).

Navier–Stokes equations for nearly incompressible flow

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu\,\triangle\boldsymbol{u} + \frac{\nu}{3}\nabla(\nabla\cdot\boldsymbol{u}) - \frac{1}{\varrho_0}\nabla p + \boldsymbol{g}$$
$$\text{or}\quad \frac{\partial u_i}{\partial t} + u_j\,u_{i|j} = \nu\,u_{i|jj} + \frac{\nu}{3}u_{j|ji} - \frac{1}{\varrho_0}p_{|i} + g_i\,, \tag{30}$$

(+ small-compressibility equation:) $$\frac{\partial p}{\partial t} = -K\,\nabla\cdot\boldsymbol{u} \quad\text{or}\quad \frac{\partial p}{\partial t} = -K\,u_{i|i}\,. \tag{31}$$

Remarks:

- These are 4 equations for 4 unknown fields: $\boldsymbol{u}$, p.
- After solution the density can be computed as $\varrho = \varrho_0\big(1 + \frac{p-p_0}{K}\big)$.

2.7 Complete Navier–Stokes equations

Mass conservation: $\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\,\nabla\cdot\boldsymbol{u} = \dfrac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\,\boldsymbol{u}) = 0.$

This is also called the *continuity equation*.

Momentum conservation: $\varrho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla\cdot\boldsymbol{\sigma} + \varrho\,\boldsymbol{g}\,,$ (here: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}$).

These are 3 equations of motion (a.k.a. balance or equilibrium equations). The symmetry of stress tensor (additional 3 equations) results from the conservation of angular momentum.

Energy conservation: $\frac{\mathrm{D}}{\mathrm{D}t}\Big(\varrho\,e + \frac{1}{2}\varrho\,\boldsymbol{u}\cdot\boldsymbol{u}\Big) = -\nabla\cdot\boldsymbol{q} + \nabla\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{u}) + \varrho\,\boldsymbol{g}\cdot\boldsymbol{u} + h.$

Here: e is the *intrinsic energy* per unit mass, $\boldsymbol{q}$ is the *heat flux vector*, and h is the *power of heat source* per unit volume. Moreover, notice that the term $\frac{1}{2}\varrho\,\boldsymbol{u}\cdot\boldsymbol{u}$ is the *kinetic energy*, $\nabla\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{u})$ is the *energy change due to internal stresses*, and $\varrho\,\boldsymbol{g}\cdot\boldsymbol{u}$ is the change of *potential energy* of gravity forces.

Equations of state and constitutive relations:

- **Thermal equation of state**: $\varrho = \varrho(p, T)$.

  For a perfect gas: $\varrho = \frac{p}{RT}$, where R is the *universal gas constant*.

- **Constitutive law for fluid**: $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u}, p) = \boldsymbol{\tau}(\boldsymbol{u}) - p\,\boldsymbol{I}$.

  For *Newtonian fluids*: $\boldsymbol{\tau} = \mu\big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}\big) - \frac{2}{3}\mu\,(\nabla \cdot \boldsymbol{u})\,\boldsymbol{I}$.

  Other relations may be used, for example: $\boldsymbol{\tau} = \boldsymbol{0}$ for an inviscid fluid, or some nonlinear relationships for non-Newtonian fluids.

- **Thermodynamic relation** for state variables: $e = e(p, T)$.

  For a calorically perfect fluid: $e = c_V\,T$, where c_V is the *specific heat at constant volume*. This equation is sometimes called the *caloric equation of state*.

- **Heat conduction law**: $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{u}, T)$.

  Fourier's law of thermal conduction with convection: $\boldsymbol{q} = -k\,\nabla T + \varrho\,c\,\boldsymbol{u}\,T$, where k is the *thermal conductivity* and c is the *thermal capacity* (the *specific heat*).

Remarks:

- There are **5** conservation equations for **14** unknown fields: ϱ, $\boldsymbol{u}$, $\boldsymbol{\sigma}$, e, $\boldsymbol{q}$.
- The constitutive and state relations provide another **11** equations and introduce **2** additional state variables: p, T.
- That gives the total number of **16** equations for **16** unknown field variables: ϱ, $\boldsymbol{u}$, $\boldsymbol{\sigma}$ (or $\boldsymbol{\tau}$), e, $\boldsymbol{q}$, p, T.
- Using the constitutive and state relations for the conservation equations leaves only **5** equations in **5** unknowns: ϱ (or p), $\boldsymbol{u}$, T.

2.8 Boundary conditions for compressible flow

Density condition:

$$\varrho = \hat{\varrho} \quad \text{on } \mathcal{S}_\varrho\,, \tag{32}$$

where $\hat{\varrho}$ is the density prescribed on the boundary.

Velocity or traction condition:

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \mathcal{S}_u\,, \qquad \text{or} \qquad \boldsymbol{\sigma}\cdot\boldsymbol{n} = \hat{\boldsymbol{t}} \quad \text{on } \mathcal{S}_t\,, \qquad \text{(or mixed)}, \tag{33}$$

where $\hat{\boldsymbol{u}}$ is the velocity vector and $\hat{\boldsymbol{t}}$ is the traction vector prescribed on the boundary.

Temperature or heat flux condition:

$$T = \hat{T} \quad \text{on } \mathcal{S}_T\,, \qquad \text{or} \qquad \boldsymbol{q}\cdot\boldsymbol{n} = \hat{q} \quad \text{on } \mathcal{S}_q\,, \qquad \text{(or mixed)}, \tag{34}$$

where $\hat{T}$ is the temperature and $\hat{q}$ is the inward heat flux prescribed on the boundary.

3 Reynolds number

Definition 0 (Reynolds number)**.** The **Reynolds number** is a dimensionless parameter defined as

$$Re = \frac{U\,L}{\nu} \tag{35}$$

where: U denotes a typical **flow speed**,
L is a characteristic **length scale** of the flow,
ν is the **kinematic viscosity** of the fluid.

The Reynolds number gives a rough indication of the **relative amplitudes of two key terms** in the equations of motion, namely,

1. the **inertial term**: $|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}| = O(U^2/L)$,
2. the **viscous term**: $|\nu\,\triangle\boldsymbol{u}| = O(\nu\,U/L^2)$.

These estimates of the order of magnitude for the inertial and viscous terms are obtained as follows.

- Derivatives of the velocity components, such as $\frac{\partial u}{\partial x}$, will typically be of order U/L, that is, the components of $\boldsymbol{u}$ change by amounts of order U over distances of order L.
- Typically, these derivatives of velocity will themselves change by amounts of order U/L over distances of order L so the second derivatives, such as $\frac{\partial^2 u}{\partial x^2}$, will be of order U/L^2.

Therefore:

$$\frac{|\text{inertial term}|}{|\text{viscous term}|} = O\left(\frac{U^2/L}{\nu\,U/L^2}\right) = O(Re)\,. \tag{36}$$

There are two extreme cases of viscous flow:

1. **High Reynolds number flow** – for $Re \gg 1$: a flow of a fluid of **small viscosity**, where **viscous effects** can be on the whole **negligible**.
   - Even then, however, viscous effects become important in **thin boundary layers**, where the unusually large velocity gradients make the viscous term much larger than the estimate $\nu\,U/L^2$. The larger the Reynolds number, the thinner the boundary layer: $\delta/L = O(1/\sqrt{Re})$ (δ – typical thickness of boundary layer).
   - A large Reynolds number is *necessary* for **inviscid theory** to apply over most of the flow field, but it is not *sufficient*.

- At high Reynolds number ($Re \sim 2000$) steady flows are often **unstable** to small disturbances, and may, as a result become **turbulent** (in fact, Re was first employed in this context).

2. **Low Reynolds number flow** – for $Re \ll 1$: a **very viscous flow**.
   - There is **no turbulence** and the flow is extremely **ordered** and nearly **reversible** ($Re \sim 10^{-2}$) – see, for example, Figure 2.

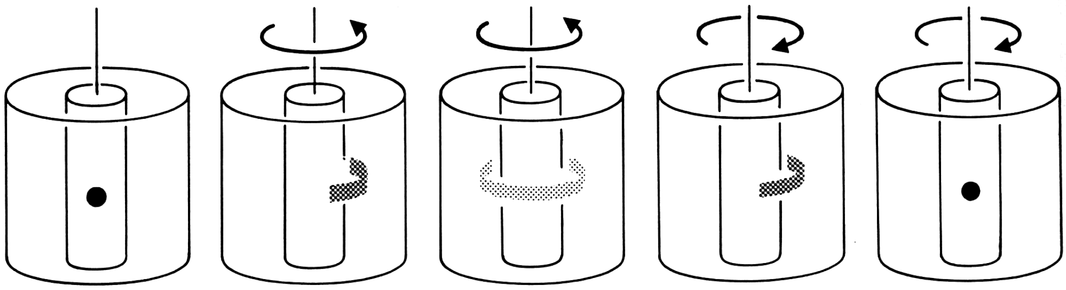

FIGURE 2: Reversibility of a very viscous flow.

4 Features of viscous flow

4.1 Viscous diffusion of vorticity

Plane parallel shear flow

$$\boldsymbol{u} = \boldsymbol{u}(y,t) = \big[u(y,t), 0, 0\big] \tag{37}$$

Such flow automatically satisfies the incompressibility condition: $\nabla \cdot \boldsymbol{u} = 0$, and in the absence of gravity the incompressibile Navier–Stokes equations of motion reduce to:

$$\frac{\partial u}{\partial t} = -\frac{1}{\varrho_0}\frac{\partial p}{\partial x} + \nu\,\frac{\partial^2 u}{\partial y^2}\,, \qquad \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0\,. \tag{38}$$

(The gravity can be ignored if it simply modifies the pressure distribution in the fluid and does nothing to change the velocity.)

- The first equation in (38) implies that $\frac{\partial p}{\partial x}$ cannot depend on x, while the remaining two equations imply that $p = p(x,t)$; therefore, $\frac{\partial p}{\partial x}$ may only depend on t.
- There are important circumstances when the flow is *not* being driven by any externally applied pressure gradient, which permits to assert that the pressures at $x = \pm\infty$ are equal. All this means that $\frac{\partial p}{\partial x} = 0$.

Diffusion equation for viscous incompressible flow

For a gravity-independent plane parallel shear flow, not driven by any externally applied pressure gradient, the velocity $u(y,t)$ must satisfy the **one-dimensional**

diffusion equation:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \,. \tag{39}$$

Example 0 (The flow due to impulsively moved plane boundary)**.**

- Viscous fluid lies at rest in the region:

  (Problem A) $0 < y < \infty$, **(Problem B)** $0 < y < h$.

- At $t = 0$ the rigid boundary at $y = 0$ is suddenly jerked into motion in the x-direction with constant speed U.
- By virtue of the no-slip condition the fluid elements in contact with the boundary will immediately move with velocity U.

▶ *Mathematical statement of the problem*

The flow velocity $u(y,t)$ must satisfy the one-dimensional **diffusion equation** (39), together with the following conditions:

1. **initial condition:**
   - $u(y,0) = 0$ (for $y \geq 0$),
2. **boundary conditions:**
   - **(Problem A)** $u(0,t) = U$ and $u(\infty,t) = 0$ (for $t \geq 0$),
   - **(Problem B)** $u(0,t) = U$ and $u(h,t) = 0$ (for $t \geq 0$).

The whole problem is in fact identical with the problem of the spreading of heat through a thermally conducting solid when its boundary temperature is suddenly raised from zero to some constant.

▶ *Solution to Problem A* (see Figure 3):

$$u = U\Big[1 - \frac{1}{\sqrt{\pi}}\int_0^{\eta} \exp\left(\tfrac{-s^2}{4}\right) \mathrm{d}s\Big] \text{ with } \eta = \tfrac{y}{\sqrt{\nu t}}, \qquad \omega = -\frac{\partial u}{\partial y} = \frac{U}{\sqrt{\pi \nu t}} \exp\left(\tfrac{-y^2}{4\nu t}\right) . \tag{40}$$

- The flow is largely confined to a distance of order $\sqrt{\nu t}$ from the moving boundary: the velocity and vorticity are very small beyond that region.
- **Vorticity diffuses** a distance of order $\sqrt{\nu t}$ in time t. Equivalently, the time taken for vosticity to diffuse a distance h is of the order $\frac{h^2}{\nu}$.

▶ *Solution to Problem B* (see Figure 4):

$$u = \underbrace{U\Big(1 - \frac{y}{h}\Big)}_{\text{steady state}} - \frac{2U}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \exp\Big(-n^2\pi^2\frac{\nu t}{h^2}\Big) . \tag{41}$$

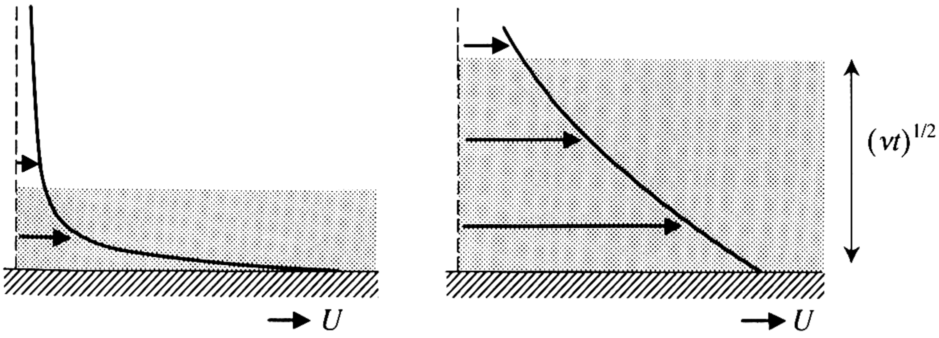

FIGURE 3: The diffusion of vorticity from a impulsively moved plane boundary: the velocity profile and the region of significant vorticity in some early and later times.

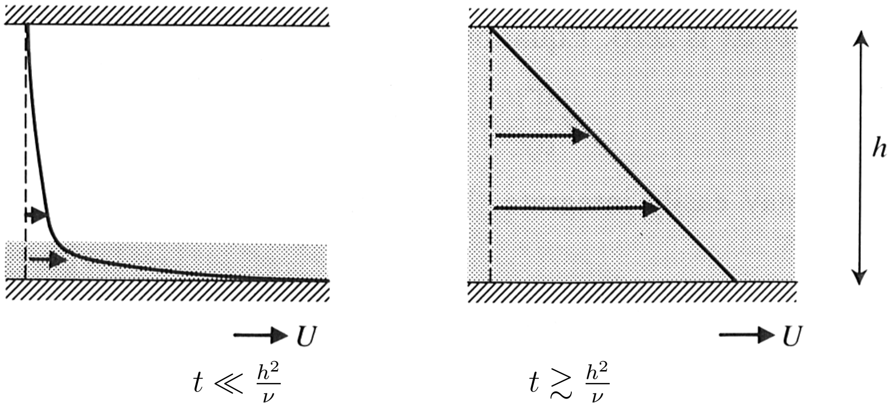

FIGURE 4: Flow between two rigid boundaries, one impulsively moved: the velocity profile and the region of significant vorticity in different times.

- For times greater than $\frac{h^2}{\nu}$ the flow has almost reached its steady state and the vorticity is almost distributed uniformly throughout the fluid.

4.2 Convection and diffusion of vorticity

Vorticity equation for viscous flows

In general:

$$\text{Incompress. Navier–Stokes} \quad \xrightarrow{\nabla\times} \quad \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} + \nu\triangle\boldsymbol{\omega}. \tag{42}$$

For a two-dimensional flow ($\boldsymbol{\omega} \perp \boldsymbol{u}$):

$$\frac{\partial \omega}{\partial t} + \underbrace{(\boldsymbol{u}\cdot\nabla)\omega}_{\text{convection}} = \underbrace{\nu\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right)}_{\text{diffusion}}. \tag{43}$$

Observation: In general, there is both **convection and diffusion of vorticity** in a viscous flow.

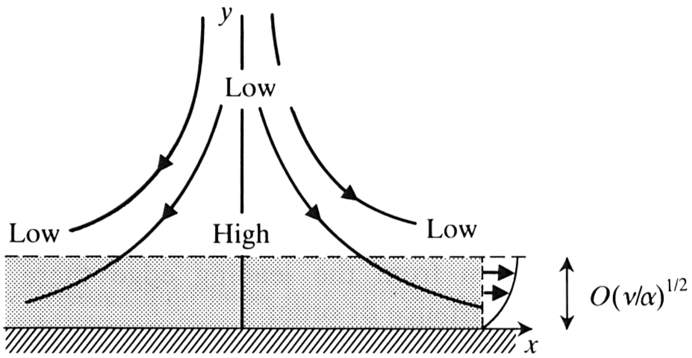

FIGURE 5: A two-dimensional flow towards a stagnation point.

Example 0 (Plane flow towards a stagnation point)**.**

Consider a plane flow towards a stagnation boundary as presented in Figure 5.

- There is an inviscid 'mainstream' flow: $u = \alpha\, x$, $v = -\alpha\, y$ (here, $\alpha > 0$ is a constant), towards a stagnation boundary at $y = 0$.
- This fails to satisfy the no-slip condition at the boundary, but the mainstream flow speed $\alpha|x|$ increases with distance $|x|$ along the boundary. By the Bernoulli's theorem, the mainstream pressure decreases with distance along the boundary in the flow direction.
- Thus, one may hope for a thin, unseparated boundary layer which adjusts the velocity to satisfy the no-slip condition.
- The boundary layer, in which all the vorticity in concentrated, has thickness of order $\sqrt{\frac{\nu}{\alpha}}$.
- In this boundary layer there is a steady state balance between the viscous diffusion of vorticity from the wall and the convection of vorticity towards the wall by the flow.
- If ν decreases the diffusive effect is weakened, while if α increases the convective effect is enhanced (in either case the boundary layer becomes thinner).

4.3 Boundary layers

- Steady flow past a fixed wing may seem to be wholly accounted for by inviscid theory. In particular, the **fluid in contact with the wing appears to slip** along the boundary.
- In fact, **there is no such slip**. Instead there is a very thin **boundary layer** (see Figure 6) where the inviscid theory fails and **viscous effects are very important**.

Boundary layer

A **boundary layer** is a very thin layer along the boundary across which the flow velocity undergoes a smooth but rapid adjustment to precisely zero (i.e. *no-slip*) on the boundary itself (see Figure 6).

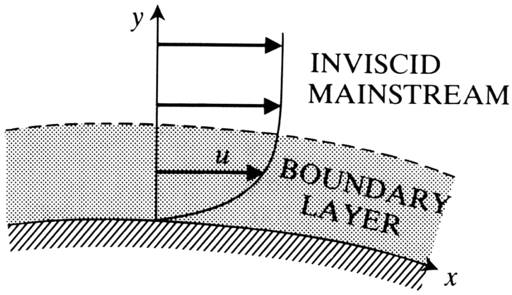

FIGURE 6: A boundary layer.

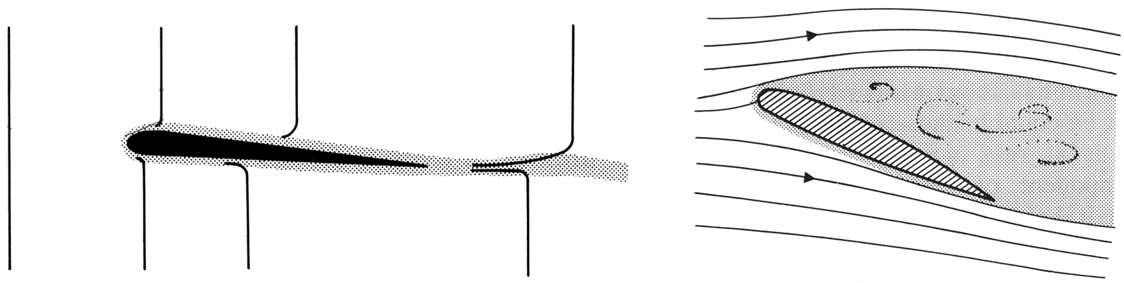

FIGURE 7: Flows past an airfoil: [left] un unseparated flow at small angle of attack (the fate of successive lines of fluid particles is shown), [right] a separated flow at large angle of attack (the layer separation is responsible for the sudden drop in lift).

- In certain circumstances **boundary layers may separate** from the boundary (see Figure 7), thus causing the whole flow of low-viscosity fluid to be quite different to that predicted by inviscid theory.
- The behaviour of a **fluid of even very small viscosity** may, on account of boundary layer separation, be **completely different** to that of a (hypothetical) **fluid of no viscosity** at all.